The Logic of Bunched Implications
A Memoir

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Abstract

This is a study of the semantics and proof theory of the logic of bunched implications (BI), which is promoted as a logic of (computational) resources, and is a foundational component of separation logic, an approach to program analysis. BI combines an additive, or intuitionistic, fragment with a multiplicative fragment. The additive fragment has full use of the structural rules of weakening and contraction, and the multiplicative fragment has none. Thus it contains two conjunctive and two implicative connectives. At various points, we illustrate a resource view of BI based upon the Kripke resource semantics. Our first original contribution is the formulation of a proof system for BI in the newly developed proof-theoretical formalism of the calculus of structures. The calculus of structures is distinguished by its employment of deep inference, but we already see deep inference in a limited form in the established proof theory for BI. We show that our system is sound with respect to the elementary Kripke resource semantics for BI, and complete with respect to the partially-defined monoid (PDM) semantics. Our second contribution is the development from a semantic standpoint of preliminary ideas for a hybrid logic of bunched implications (HBI). We give a Kripke semantics for HBI in which nominal propositional atoms can be seen as names for resources, rather than as names for locations, as is the case with related proposals for BI-Loc and for intuitionistic hybrid logic. The cost of this approach is the loss of intuitionistic monotonicity in the semantics. But this is perhaps not such a grave loss, given that our guiding analogy is of states of models with resources, rather than with states of knowledge, as is standard for intuitionistic logic.

Declaration

I declare that:

(i) This thesis comprises only my original work, except where due acknowledgement has been made in the Preface;

(ii) Due acknowledgement has been made in the text to all other material used;

(iii) This thesis contains no work used by me for the award of another degree;

(iv) This thesis is approximately 30,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Benjamin Horsfall
June 12, 2007

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Contents

List of Figures

Preface

This thesis gives a presentation of, and a commentary upon the propositional logic of bunched implications (BI) [O'Hearn & Pym 1999, Pym 2002] from proof-theoretical and semantic points of view. Although the thesis is for the most part self-contained, it is expected that the reader will have some background knowledge of structural proof theory — or more particularly of sequent calculi for intuitionistic or linear logic — and of Kripke's possible worlds semantics for intuitionistic or modal logic. It presents and makes use of the formulations of Kripke resource semantics for BI given by Pym [2002], Pym, O'Hearn & Yang [2004] and particularly Galmiche, Méry & Pym [2005]. In addition, it presents two original contributions.

The first is the system SBISg, a formulation of BI in the proof-theoretical formalism of Guglielmi's [2004] calculus of structures, together with detailed soundness and completeness proofs. The formulation here is based upon Tiu's [2005, 2006] work on intuitionistic logic in the calculus of structures. The soundness proof is along conventional lines, except that it requires an original 'semantic depth' lemma to handle deep inference in SBISg. The completeness proof is indebted in its strategy and techniques to completeness proofs for BI given by Pym [2002], Pym, O'Hearn & Yang [2004] and and Galmiche, Méry & Pym [2005]; for intuitionistic logic given by van Dalen [2004]; and for modal logic given by Blackburn, de Rijke & Venema [2001]. Completeness is shown with respect to a variant of the partially-defined monoid (PDM) semantics [Galmiche, Méry & Pym 2005, §5.3]. My paper [Horsfall 2006] presenting the formulation of BI in the calculus of structures was accepted for the 11th ESSLLI Student Session at the 18th European Summer School in Logic, Language and Information, July–August 2006, but I could not attend to present it, and consequently, it had to be withdrawn. This paper has been incorporated into the introduction and §§1.2–3.3.

The second contribution is the development from a semantic standpoint of preliminary ideas for a hybrid logic of bunched implications, based upon established ideas on hybrid modal logics [see, for instance, Blackburn 2000b, Blackburn, de Rijke & Venema 2001] and hybrid intuitionistic logics [Jia & Walker 2004, Braüner & de Paiva 2006, Chadha, Macedonio & Sassone 2006]. The ideas developed here differ significantly from BI-Loc [Biri & Galmiche 2003] and those in the literature on hybrid logics.

Chapter 1

Introduction

1.1. The logic of bunched implications

The logic of bunched implications (BI) [O'Hearn & Pym 1999, Pym 2002] permits control of the structural rules of weakening and contraction – familiar from sequent calculi for many systems – in a way quite different to linear logic. In linear logic, ordinary conjunction decomposes into two distinct connectives when weakening and contraction are not available, yielding *multiplicative* or *context-free* conjunction $\otimes$ and disjunction $\invamp$, and *additive* or *context-sharing* & and $\oplus$. The availability of structural rules may then be selectively granted using exponential operators ? and !. Something similar occurs in BI, except that there are no exponentials. Weakening and contraction are always available in additive contexts, and never in multiplicative ones.

Definition 1 The set Φ of formulæ of propositional BI is given by the grammar:

$$\phi ::= p \mid I \mid \phi * \phi \mid \phi \mathrel{-\!\!*} \phi \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi$$

where $p, q, r, \ldots \in \mathcal{P}$ are propositional variables, $*$ is multiplicative conjunction, the propositional constant I its unit, $\mathrel{-\!\!*}$ multiplicative implication, and the remainder the usual intuitionistic connectives and propositional constants. $\square$

There is no multiplicative *falsum*, and the only available negation is intuitionistic, definable $\neg\phi := \phi \to \bot$, which is in any case not involutive. Bunches, the structured antecedents of the sequent calculus LBI[1] are really an artefact of the calculus. Bunches are tree structures.

Definition 2 Bunches are given by the grammar:

$$\Gamma ::= \phi \mid \varnothing_m \mid \Gamma\,,\Gamma \mid \varnothing_a \mid \Gamma\,;\Gamma$$

"," is the multiplicative bunch constructor, ";" the additive constructor, and $\varnothing_m$ and $\varnothing_a$ their respective units. □

Definition 3 Δ is a *sub-bunch* of Γ if Γ and Δ are the same bunch, or if Γ has the structure $\Gamma_1\,;\Gamma_2$ or $\Gamma_1\,,\Gamma_2$ and Δ is a sub-bunch of either Γ_1 or Γ_1. When Δ is a sub-bunch of Γ, we can write Γ as $\Gamma(\Delta)$ to express this fact, and to pick out the *bunched context* $\Gamma(.)$, which is an incomplete bunch the same as Γ, except with a gap in it where the sub-bunch Δ was. The bunch obtained by replacing the sub-bunch Δ of Γ with a new sub-bunch Δ' is written $\Gamma[\Delta'/\Delta]$. More often, we write $\Gamma(\Delta')$, as in the specification of the rules of inference of LBI in Figure 1.1. □

Definition 4 Equivalence ($\equiv$) of bunches, also called *coherent equivalence*, is modulo commutativity of "," and ";", combination with their respective units, and congruence, that is $\Gamma(\Delta) \equiv \Gamma(\Delta')$ if $\Delta \equiv \Delta'$. □

The left-hand rules of LBI may match and manipulate sub-bunches at any depth in a bunch. Cut-elimination holds for LBI [Pym 2002, Theorem 6.2]. That is, if a sequent is provable in LBI, then it is provable in LBI without the *cut* rule.

In Girard's idiom, "," is 'hypocrisy' for $*$, ";" for $\wedge$, $\varnothing_m$ for I and $\varnothing_a$ for $\top$. This distinction between punctuation marks and logical connectives does not figure directly in the categorical view, and is abolished in the calculus of structures. A sequent of LBI has the form $\Gamma \Rightarrow \phi$. As in typical intuitionistic sequent calculi, the succedent is restricted to a single formula. In fact, formulations of intuitionistic logic generally restrict it to *at most* one formula. The desired effect of the

[1]See Figure 1.1 for the rules of inference of LBI. Pym's monograph [2002, chapter 6] with errata [2006] gives a full treatment of the sequent calculus LBI for propositional BI. Figure 1.2 reproduces a proof of the interesting theorem $((p \to q) \mathrel{-\!\!*} r) \mathrel{-\!\!*} (p \to (q \mathrel{-\!\!*} r))$ in the sequent calculus LBI, found by Alwen Tiu (personal communication). Note that the converse formula $(p \to (q \mathrel{-\!\!*} r)) \mathrel{-\!\!*} ((p \to q) \mathrel{-\!\!*} r)$ is not a theorem.

$$\frac{}{\phi \Rightarrow \phi}\ \textit{identity} \qquad \frac{\Delta \Rightarrow \phi \qquad \Gamma(\phi) \Rightarrow \psi}{\Gamma(\Delta) \Rightarrow \psi}\ \textit{cut} \qquad \frac{\Gamma \Rightarrow \phi}{\Gamma' \Rightarrow \phi}\ \textit{exchange}\ (\Gamma \equiv \Gamma')$$

$$\frac{\Gamma(\Delta) \Rightarrow \phi}{\Gamma(\Delta\,;\Delta') \Rightarrow \phi}\ \textit{weakening} \qquad \frac{\Gamma(\Delta\,;\Delta) \Rightarrow \phi}{\Gamma(\Delta) \Rightarrow \phi}\ \textit{contraction}$$

$$\frac{}{\Gamma(\bot) \Rightarrow \phi}\ \bot_L \qquad \frac{\Gamma(\varnothing_m) \Rightarrow \phi}{\Gamma(I) \Rightarrow \phi}\ I_L \qquad \frac{}{\varnothing_m \Rightarrow I}\ I_R \qquad \frac{\Gamma(\varnothing_a) \Rightarrow \phi}{\Gamma(\top) \Rightarrow \phi}\ \top_L \qquad \frac{}{\varnothing_a \Rightarrow \top}\ \top_R$$

$$\frac{\Gamma(\phi\,,\psi) \Rightarrow \chi}{\Gamma(\phi * \psi) \Rightarrow \chi}\ *_L \qquad \frac{\Gamma \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Gamma\,,\Delta \Rightarrow \phi * \psi}\ *_R \qquad \frac{\Delta \Rightarrow \phi \qquad \Gamma(\psi\,,\Delta') \Rightarrow \chi}{\Gamma(\Delta\,,\phi \mathrel{-\!\!*} \psi\,,\Delta') \Rightarrow \chi}\ \mathrel{-\!\!*}_L \qquad \frac{\Gamma\,,\phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \mathrel{-\!\!*} \psi}\ \mathrel{-\!\!*}_R$$

$$\frac{\Gamma(\phi\,;\psi) \Rightarrow \chi}{\Gamma(\phi \wedge \psi) \Rightarrow \chi}\ \wedge_L \qquad \frac{\Gamma \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Gamma\,;\Delta \Rightarrow \phi \wedge \psi}\ \wedge_R \qquad \frac{\Delta \Rightarrow \phi \qquad \Gamma(\psi\,;\Delta') \Rightarrow \chi}{\Gamma(\Delta\,;\phi \to \psi\,;\Delta') \Rightarrow \chi}\ \to_L \qquad \frac{\Gamma\,;\phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi}\ \to_R$$

$$\frac{\Gamma(\phi) \Rightarrow \chi \qquad \Gamma(\psi) \Rightarrow \chi}{\Gamma(\phi \vee \psi) \Rightarrow \chi}\ \vee_L \qquad \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi}\ \vee_{R_{left}} \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi}\ \vee_{R_{right}}$$

Figure 1.1: The sequent calculus LBI

restriction is to prevent contraction on the right-hand side. This prevents, for instance, proofs of $\Rightarrow\ p \vee \neg p$ and $\neg\neg p\ \Rightarrow\ p$. The restriction was discovered by Gentzen [1934–35]. See, for instance, Troelstra & Schwichtenberg [2000, §3.1]. There are, of course, other ways to obtain the same effect. Obviously, we have linear logic, but also for instance, Dragalin's multisuccedent intuitionistic system [see Negri & von Plato 2001, §5.3]. In this system, we find the rule $\frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi, \Delta} \to_R$. Restriction of the succedent occurs in this rule. The arbitrary context Δ in the conclusion is forcibly weakened away. Weakening and contraction are admissible in general, but implication is forced through a 'bottleneck' which is sufficient to keep the system intuitionistic. Tiu [2005] makes a similar consideration.

It is a noteworthy characteristic of LBI that corresponding pairs of additive and multiplicative rule are 'morphologically' similar. Take, for example, the right-side rules for additive and multiplicative conjunction, respectively:

$$\frac{\Gamma \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Gamma\,;\Delta \Rightarrow \phi \wedge \psi}\ \wedge_R \qquad \frac{\Gamma \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Gamma\,,\Delta \Rightarrow \phi * \psi}\ *_R$$

The only point of structural difference is the bunch constructor in each case, respectively additive ";" and multiplicative ",". It may be surprising to observe that these are both 'context-splitting' rules. We are used to the idea that additive rules (or connectives) are context-sharing and multiplicative rules are context-splitting (or context-free). For instance, this is precisely the difference between the sequent rules for additive and multiplicative conjunction in linear logic:

$$\frac{\Rightarrow \Gamma, \phi \qquad \Rightarrow \Gamma, \psi}{\Rightarrow \Gamma, \phi \,\&\, \psi}\ \textit{with} \qquad \frac{\Rightarrow \Gamma, \phi \qquad \Rightarrow \psi, \Delta}{\Rightarrow \Gamma, \phi \otimes \psi, \Delta}\ \textit{times}$$

These connectives are on equal footing when it comes to the availability of structural rules. If we think for the moment in terms of upward proof-search, the difference is just that the rules are applicable at different types of splitting-points in a bunch. These right-side rules are only applicable at the top-level bunch constructor, but the corresponding left-side rules are cases of deep inference, which is employed more systematically in the calculus of structures. Consider the left-side rule for multiplicative implication:

$$\frac{\Delta \Rightarrow \phi \qquad \Gamma(\psi\,, \Delta') \Rightarrow \chi}{\Gamma(\Delta\,, \phi \mathrel{-\!\!*} \psi\,, \Delta') \Rightarrow \chi} \mathrel{-\!\!*}_L$$

which says that if Δ 'entails' ϕ and Γ 'entails' χ, and ψ, Δ' is a sub-bunch of Γ at arbitrary depth (which multiplicatively combines a formula ψ and any bunch Δ') then χ is 'entailed' by the bunch formed by replacing ψ, Δ' in Γ with a new multiplicatively constructed sub-bunch $\Delta, \phi \mathrel{-\!\!*} \psi, \Delta'$.

The real structural difference between the additive and multiplicative fragments of LBI is the applicability of structural rules. Weakening is only available at additive splitting-points, again, of sub-bunches at arbitrary depth, and contraction may only construct additively-combined duplicate sub-bunches. The morphological similarity of the multiplicative and non-structural additive fragments, together with limitation on the availability of structural rules, will be apparent in our initial formulation of BI in the calculus of structures in Chapter 3.

1.2. THE CALCULUS OF STRUCTURES

Proof systems in the calculus of structures have a single object – a *structure* – in place of formulæ and sequents. This is perfectly natural, given that a sequent may typically be encoded as a single formula. For instance, in linear logic [Girard 1987a], a one-sided sequent $\Rightarrow \phi, \psi$ may be encoded as $\phi \parr \psi$. In a classical sequent calculus $\phi, \psi \Rightarrow \chi, \zeta$ may be encoded as $\phi \wedge \psi \rightarrow \chi \vee \zeta$. The calculus of structures, invented by Guglielmi [2004], has been used to

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{q \Rightarrow q}\,\mathit{identity}}{q\,;p \Rightarrow q}\,\mathit{weakening}}{q \Rightarrow p \to q}\,\to_R \qquad \cfrac{}{r \Rightarrow r}\,\mathit{identity}}{(p \to q) \mathrel{-\!\!*} r\,, q \Rightarrow r}\,\mathrel{-\!\!*}_L}{(p \to q) \mathrel{-\!\!*} r \Rightarrow q \mathrel{-\!\!*} r}\,\mathrel{-\!\!*}_R}{(p \to q) \mathrel{-\!\!*} r\,; p \Rightarrow q \mathrel{-\!\!*} r}\,\mathit{weakening}}{(p \to q) \mathrel{-\!\!*} r \Rightarrow p \to (q \mathrel{-\!\!*} r)}\,\to_R}{\varnothing_m \Rightarrow ((p \to q) \mathrel{-\!\!*} r) \mathrel{-\!\!*} (p \to (q \mathrel{-\!\!*} r))}\,\mathrel{-\!\!*}_R$$

Figure 1.2: A proof of the theorem $((p \to q) \mathrel{-\!\!*} r) \mathrel{-\!\!*} (p \to (q \mathrel{-\!\!*} r))$ in LBI.

formulate various logical systems, for instance: multiplicative exponential linear logic [Straßburger 2003] and other varieties of linear logic, classical logic [Brünnler 2006] and intuitionistic logic [Brünnler 2004, Tiu 2005].

The approach appears to have been most fruitful for systems with an involutive negation (i.e. An operator $\neg$ such that $\neg\neg\phi \equiv \phi$) and de Morgan duality – classical and linear logic. These systems are also symmetric, in the sense that they admit sequents with multiple conclusions (a comma in the succedent being a disjunction of some sort), and involutive negation allows a free traffic of formulæ between antecedent and succedent. Difficulties arise when these characteristics are absent, as in intuitionistic logic and BI. In particular, it is necessary to introduce a notion of *polarity* to do some of the work of involutive negation. Essentially, polarity amounts to a 'sidedness' annotation restricting the applicability of rules, and corresponds to the division of left- and right-side rules in sequent calculi.

The calculus of structures differs from sequent calculi in several respects. In particular, rules of inference may be applied at arbitrary depth within a structure, unlike most sequent calculi where only the outermost connective of a formula is available to be matched with a rule. This is called *deep inference*. A significant feature of the sequent calculus LBI – which makes it unusual amongst sequent calculi in general – is that employs a limited form of deep inference, whith its left-hand rules applicable at arbitrary depth within a bunch. In a calculus of structures each rule has exactly one premise, so proofs do not have the branching structure of proofs in a sequent calculus. This removes one source of indeterminacy found

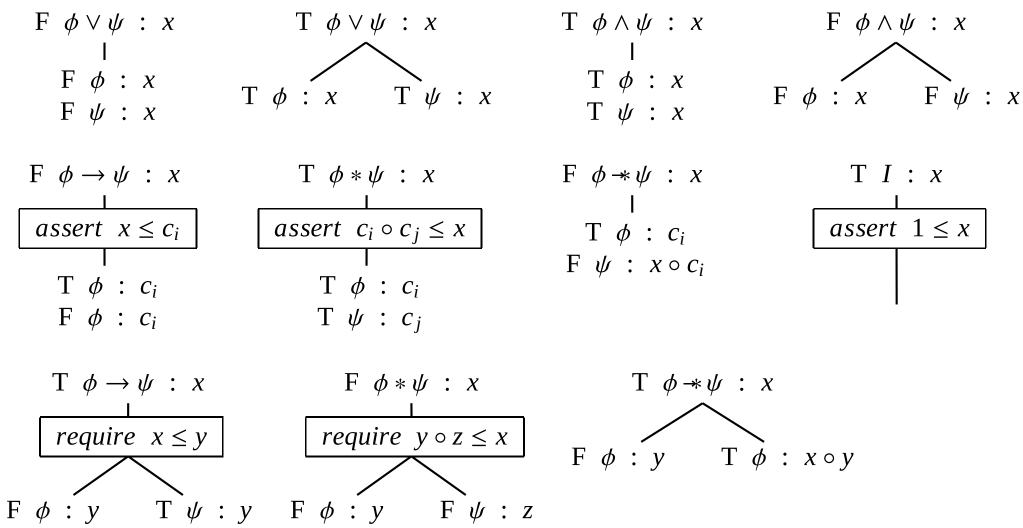

Figure 1.3: The tableau expansion rules of TBI

in sequent calculi: consider the $\otimes$ rule of linear logic which needs to partition the context of a conclusion between two premises. Of course, deep inference will be a source of indeterminacy for any proof-search, as it for proof-search in LBI. And branching in proofs, even without indeterminacy, makes the invertibility of rules awkward. In the calculus of structures, proofs exhibit an up-down symmetry, with each rule having a 'contrapositive' dual, or *corule*. The use of single-premise rules, together with deep inference, makes the calculus of structures a term rewriting system.

1.3. RESOURCE TABLEAUX

Galmiche et al. [2005] have developed *resource tableaux*, a semantic tableaux proof system for BI, in several variants. Resource tableaux are closely connected to their work on the revised semantics for BI, which we treat in Chapter 2, and yield decidability and the finite model property for BI [Galmiche et al. 2005, §8]. Resource tableaux are particularly suitable as a basis for theorem-proving implementations for BI. They are certainly more suitable than than the sequent calculus LBI or the system in the calculus of structures which we present in Chapter 3, which due to their shared characteristic of deep inference, suffer the high degree of indeterminacy in proof-search remarked above, in addition to well-known issues surrounding the control of contraction rules. We have used TBI as the basis for a successful theorem prover for propositional BI implemented in Haskell, which renders tableaux using LaTeX.[2] Figure 1.3 reproduces the tableau expansion rules for Galmiche, Méry & Pym's [2005] resource tableaux system TBI.[3] In the expansion rules, x, y, z are variables over labels, and c_i, c_j are fresh labels introduced into a tableau by an expansion rule. The *assert* boxes assert constraints upon labels, and the *require* boxes require that existing labels in a tableau meet a specified constraint for an expansion rule to be applicable. Labels and constraints upon them generate a labelling algebra as the closure under certain conditions [see Galmiche et al. 2005, Definition 3.3] of a set of constraints. A labelling algebra has precisely the same structure as a model in the Kripke resource semantics (see Chapter 2), and contains a preorder $\leq$ and binary combination operator $\circ$. Also note the correspondence between the expansion rules and the clauses of the forcing relation in the Kripke resource semantics. The root formula of a tableau is assigned a polarity F and labelled **1**, which is the unit label for $\circ$, and it is proven if it can be expanded to a tableau in which every branch is closed. We can construct a 'resource' dependency graph of labels as we construct a tableau. The condition for a tableau branch to be closed is more complicated than simple contradiction [see Galmiche et al. 2005, Definition 3.8]. A countermodel for a formula may be constructed from an open branch of a tableau for that formula [Galmiche et al. 2005, §4.2], and this is the key to decidability. Figure 1.4 presents a proof in TBI of the theorem that was proven using LBI in Figure 1.2. By way of example, observe that the rightmost branch is closed because it contains the signed and labelled occurences $\mathrm{T}\ r\ :\ c_1 \circ c_3$ and $\mathrm{F}\ r\ :\ c_2 \circ c_3$ such that $c_1 \circ c_3 \leq c_2 \circ c_3$, by the bifunctoriality property of the labelling algebra: if $x \leq y$ then $x \circ z \leq y \circ z$.

[2]See also BILL [Béal, Méry & Galmiche n.d., Galmiche & Méry 2004], a tableaux-based theorem prover for propositional BI without propositional constants, implemented in Objective Caml.

[3]See also the treatment of liberalised TBI' [Galmiche et al. 2005, §§6, 7].

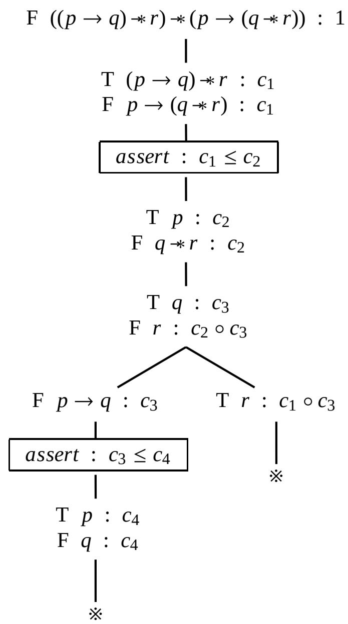

Figure 1.4: A proof of the theorem $((p \to q) \mathrel{-\!*} r) \mathrel{-\!*} (p \to (q \mathrel{-\!*} r))$ in TBI.

1.4. A CATEGORICAL VIEW OF PROOFS IN BI

We may characterise BI using the categorical model for proofs in BI, given by Pym and O'Hearn.[4] This kind of view is sometimes called a "categorical semantics", but it is more an abstract view of proof structures than a semantics. In categorical logic, generally speaking, we view formulæ as *objects* in categories and proofs as *arrows* in those categories. A pair of arrows to and from an object may always be composed to form another arrow, yielding a sort of transitivity of proofs, or cut. Proofs of BI are modelled using *doubly closed categories* (DCCs). A DCC is a 'superimposition' of two *closed categories* in a single category, with one of these being *cartesian*.[5] A closed category $\mathcal{C}$ is a symmetric monoidal category[6] in which every functor[7] $- \otimes B : \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint $(-)^B : \mathcal{C} \longrightarrow \mathcal{C}$. In

[4]See O'Hearn & Pym [1999, §3] and Pym [2002, §3.3]. We refer to Mac Lane [1998] as a reference on categorical concepts.

[5]I closely follow the presentation of CCCs in Lambek & Scott [1986, §§I.1–3, 8].

[6]A monoidal category has a product operation, or *bifunctor* $\otimes$ that is associative and has (the same) right and left unit (up to isomorphism). It is symmetric if the bifunctor is commutative.

[7]A functor is a morphism between categories, i.e. a function from each object of one to an object of the other, and a function likewise for arrows. We may obtain a functor $- \otimes B$ by 'partial

BI, $*$ and $\wedge$ are the product operations of the closed categories, and right adjoints are constructed from the two conditionals. The adjunctions are characterised by natural isomorphisms of hom-sets (i.e. sets of arrows from one object to another): $[A * B, C] \cong [A, B \mathrel{-\!\!*} C]$ and $[A \wedge B, C] \cong [A, B \to C]$. Note that these are incarnations of currying. An important isomorphism arising in the closed categories is $[I, A \mathrel{-\!\!*} B] \cong [A, B] \cong [\top, A \to B]$ which is the special case for units. This does, not, however entail the equivalence of the two implications. In general $\phi \mathrel{-\!\!*} \psi \nvdash \phi \to \psi$ and $\phi \to \psi \nvdash \phi \mathrel{-\!\!*} \psi$. These isomorphisms are important points of reference when we define the syntactic equality of structures in Chapter 3.

A cartesian closed category (CCC) is a closed category in which the product is cartesian ($\times$): it has all finite products given. Given arrows $f : C \longrightarrow A$ and $g : C \longrightarrow B$ in a CCC, there exists a unique arrow $\langle f, g \rangle : C \longrightarrow A \times B$. Arrows in a CCC satisfy certain requirements:

(i) $\forall A \exists f \,.\, f : A \longrightarrow \top$ ($\top$ is the *terminal object*)

(ii) $\pi_{A,B} \circ \langle f, g \rangle = f$

(iii) $\pi'_{A,B} \circ \langle f, g \rangle = g$

The arrows $\pi_{A,B} : A \times B \longrightarrow A$ and $\pi'_{A,B} : A \times B \longrightarrow B$ are projections, and exist for all objects A, B in a CCC. In fact, the CCC for BI is *bicartesian*, meaning that it also has all finite *coproducts* ($+$, in our case $\vee$) and an initial object ($\perp$). For each $f : A \longrightarrow C$ and $g : B \longrightarrow C$, there is a unique arrow $[f, g] : A + B \longrightarrow C$. Arrows meet the further requirements:

(i) $\forall A \exists f \,.\, f : \perp \longrightarrow A$

(ii) $[f, g] \circ \kappa_{A,B} = f$

(iii) $[f, g] \circ \kappa'_{A,B} = g$

The arrows $\kappa_{A,B} : A \longrightarrow A + B$ and $\kappa'_{A,B} : B \longrightarrow A + B$ are injections and exist for all A, B.

We can give a categorical justification of the structural rules weakening and contraction in a CCC. Remember that ";" is just a cipher for $\wedge$. Let $\mathcal{C}$ be a CCC.

application' of a bifunctor.

(i) *Weakening.* Consider objects $A, C \in \mathcal{C}$ and an arrow $f : A \longrightarrow C$ of $\mathcal{C}$, that is to say, a proof of C from A. Since $\mathcal{C}$ is a CCC, there exists an arrow $\pi_{A,B} : A \times B \longrightarrow A$ for any $B \in \mathcal{C}$. So by composition, the arrow $f \circ \pi_{A,B} : A \times B \longrightarrow C$ exists, and hence there is a proof of C from $A \wedge B$. Similarly, $\pi'_{B,A} : B \times A \longrightarrow A$ exists, and by composition we have $f \circ \pi'_{B,A} : B \times A \longrightarrow C$, that is, a proof of C from $B \wedge A$.

(ii) *Contraction.* Consider objects $A, B \in \mathcal{C}$ and an arrow $f : A \times A \longrightarrow B$, that is, a proof of B from $A \wedge A$. There exists an identity arrow $1_A : A \longrightarrow A$, and since $\mathcal{C}$ is a CCC, there exists an arrow $\langle 1_A, 1_A \rangle : A \longrightarrow A \times A$. Composition with f yields the arrow $\langle 1_A, 1_A \rangle \circ f : A \longrightarrow B$, which is a proof of B from A.

A model $\langle \mathcal{C}, V \rangle$ for BI is a bicartesian DCC $\mathcal{C}$ together with a valuation function $V : \mathcal{P} \to O$ from propositional variables to objects $O \in \mathcal{C}$. We may then inductively define a function $V^* : \Phi \to O$ from (atomic and compound) formulæ to $O \in \mathcal{C}$ according to the preceding sketch. We encode of a bunch Γ into a formula γ according to the obvious translation. Given a sequent $\Gamma \Rightarrow \phi$, we then ask: does there exist an arrow $V^*(\gamma) \longrightarrow V^*(\phi)$? Each arrow is proof. (Then the question of the identity of proofs becomes the question of the identity of arrows.) If there are no arrows, the sequent is not provable. There is a hint here for our formulation in the calculus of structures: we do not encode the $\Rightarrow$, so we do not have to choose between encoding it using either as intuitionistic ($\to$) or multiplicative ($-\!\!*$) implication. Instead, we ask "Is there an arrow?" We can think of this from the point-of-view of theoremhood in LBI. ϕ is a theorem iff either $\varnothing_a \Rightarrow \phi$ or $\varnothing_m \Rightarrow \phi$ is provable. (Since $\varnothing_m \Rightarrow \phi$ may be derived from $\varnothing_a \Rightarrow \phi$ using weakening, it suffices that $\varnothing_m \Rightarrow \phi$.) $\phi \to \psi$ is a theorem iff $\phi -\!\!* \psi$ is a theorem. Of course, that this does not amount to equivalence in general. So attempting to prove $\Gamma \Rightarrow \phi$ is equivalent to testing the theoremhood of either $\gamma \to \phi$ or $\gamma -\!\!* \phi$. So when we encode $\Rightarrow$ in a structure, we are allowed an arbitrary choice.

The principal point of this categorical view is that although there are two distinct conjunctions in BI, each with an adjoint implication – and that in this respect they are structurally the same – only the additive (or intuitionistic) CCC structure is granted the use of weakening and contraction.

1.5. APPLICATIONS OF BI

BI may be seen broadly as a logic for reasoning about (computational) resources. It has been proposed as the basis of a type theory for (imperative) programs, by way of a Curry-Howard correspondence, to control sharing and non-sharing of data and other resources by components of programs [O'Hearn 2003]. For instance, $a \mathrel{-\!\!*} b$ might be specified as the type of a function, as a constraint (or a guarantee) that in its internal workings, the function not share memory or some other sort of resource with its argument. Similarly, $a * b$ might be given as the type of a tuple of objects which are disjoint in their use of resources. On the other hand, $a \to b$ would be the commonplace intuitionistic function type, and $a \wedge b$ the tuple type, familiar from functional programming, and which do not constrain this kind of sharing. In typical functional programming settings, the question of this sort of sharing between functions and arguments does not arise, because of the characteristic referential transparency of functions. BI also plays an important foundational rôle in the research program of separation logic. Separation logic has been used to analyse the shared use of mutable data structures by imperative programs [Ishtiaq & O'Hearn 2001, O'Hearn, Reynolds & Yang 2001, Reynolds 2002] and resource use by concurrent programs [O'Hearn 2005]. Armelín & Pym [2001] develop a logic programming language BLP based upon BI which manages sharing and non-sharing of resources.

1.6. OVERVIEW

In Chapter 2, we present a survey of various refinements of the Kripke resource semantics for BI, and its predecessors: Saul Kripke's semantics for intuitionistic logic and Alasdair Urquhart's semantics for relevant logics. We make some refinements, which are technically motivated. This lays the foundation for the main contribution, contained in Chapter 3. We present a proof system for BI in the emerging proof-theoretical formalism of the calculus of structures, and give proofs of its soundness and completeness. This work is a theoretical and technical contribution to two distinct research programs: the logical theory – both the proof theory and the semantics – of BI, and the calculus of structures. We provide evidence for the naturalness, versatility and flexibility of the formalism of the calculus of structures, in particular addressing problems with the treatment of logics of the intuitionistic family, and logics containing more than one kind of implication. Admittedly, the calculus of structures does not represent any great advance in automated theorem proving for BI – resource tableaux are best for that – but it does permit a presentation of BI and a method of proof that is arguably simpler, more direct and more natural than with other formalisms, capturing the essential structure of this notoriously complicated logic in an intuitively satisfying way. Deep inference, which is a natural characteristic of BI, is generalised in the calculus of structures, and the new formalism permits a view much more faithful to the powerfully intuitive categorical view of BI as a bicartesian DCC. This work benefits BI research by demonstrating the naturalness and generality of BI in its adaptation to the new formalism, in the process bringing some of the more subtle characteristics of BI to the fore. In Chapter 4 we undertake a tentative and speculative exploration of the idea of a hybrid BI, introducing ideas from the research program of hybrid modal logics (which have applications to distributed computation, for example). Our principal motivation is to introduce *names for resources* into propositional BI. Although our proposal is quite abstract, we believe that it might turn out to improve upon the expressiveness of BI for reasoning about resource distribution, by introducing into the language the ability to name certain resources in a system.

A sequence of informal, practical reflections runs through the thesis. The reader is referred particularly to the philosophical remarks upon Urquhart's semantics and Kripke's intuitionistic semantics in §2.1 and §2.2, and the reflections on the idea of 'states as resources' in §2.4 and on combining hybrid and resource logics in §4.3.

Chapter 2

The semantics of BI

Kripke semantics are natural and appealing for intuitionistic, modal and hybrid logics because they represent relational structures. In the intuitionistic case, they give a natural representation of epistemic progress as a tree structure. They provide highly intuitive models for situated, or local reasoning, and for the traversal of relational structures by agents. Kripke semantics give accounts of the meanings of statements, which may or may not be valid in every model, at particular positions in a relational structure. Proof theory, on the other hand, concerns itself with provability, in particular, the provability of *theorems*, and with the structures of proofs. Provability is the syntactic analogue of validity in every model. Proof theory does not concern itself so much with non-theorems, or contingent propositions, except as structural elements of proofs.[1]

We would like to obtain a clear understanding of BI as a logic of resources by way of an examination of its semantics. In particular, we will examine the roots of the Kripke resource semantics in Kripke's [1965] possible worlds semantics for intuitionistic logic and in Urquhart's [1972] semantics for relevant logic. We will look at the way in which these have been combined to produce the semantics of BI. This chapter gives a survey of the possible worlds semantics, or Kripke resource semantics of BI that are developed in several variations in O'Hearn & Pym [1999], in Pym's monograph [2002, 2006], in Pym, O'Hearn & Yang [2004]

[1]The research program of proof-theoretic semantics [see, for instance, Prawitz 2006] is an exception here, with the attention it gives to Gentzen's [1934–35] idea that the meanings of logical connectives be understood by way of their introduction rules in natural deduction systems.

and Galmiche, Méry & Pym [2005].[2] We confine our attention to propositional BI.

There are several Kripke semantics for BI. Under the simplest formulation, BI without the propositional constant $\perp$ is sound and complete with respect to the to the *elementary* Kripke resource semantics [Pym 2002, §4.2].[3] The soundness and completeness results in Pym [2002] are actually proven directly for the natural deduction system NBI of BI. Results for LBI are furnished by proof-theoretical equivalence with NBI. BI with $\perp$ is sound and complete with respect to several kinds of topological semantics, including Grothendieck sheaf-theoretic semantics, also called Grothendieck resource semantics [Pym, O'Hearn & Yang 2004, Pym 2002, chapter 5]. We will not treat the topological semantics, but instead some other more recently developed semantics: the *new* relational semantics, the *new* Kripke resource semantics, and the partially-defined monoid (PDM) semantics. These three semantics are closely related and are presented by Galmiche, Méry & Pym [2005, §§5.1, 5.2, 5.3 respectively]. These accounts of the semantics are closed entwined with the work on semantic tableau proof methods for BI by Galmiche & Méry [2001, 2003, 2005] and by Galmiche, Méry & Pym [2002, 2005]. Galmiche et al. [2005] give detailed soundness (Theorem 5.1) and completeness (Theorem 5.2) proofs for LBI[4] with respect to the new relational semantics. The new Kripke semantics is the special case of the new relational semantics in which $x \bullet y \sqsubseteq z$ is defined as $R_{\sqsubseteq}xyz$.[5] This fact is used to prove the soundness of LBI with respect to the new Kripke semantics [Galmiche et al. 2005, Theorem 5.3]. Completeness of LBI with respect to the new Kripke semantics can be shown using the equivalence of the Kripke resource models with Grothendieck resource models [Galmiche et al. 2005, Lemma 5.6], and the known completeness of LBI with respect to the Grothendieck resource semantics [Galmiche et al. 2005, Theorem 2.5].[6] Hence we have the equivalence of the new relational semantics and the

[2] See also Pym's conference paper [1999] on predicate BI.

[3] See also Galmiche et al. [2005, §2.1].

[4] Galmiche et al. frequently write BI when they mean to refer to the proof system LBI, which is sometimes a little confusing, although TBI is always referred to explicitly.

[5] See §2.6 below and Galmiche et al. [2005, p. 1067].

[6] Cf. the proof of Theorem 5.7 in Galmiche et al. [2005] which invokes a tableau-based countermodel construction technique and the completeness of the tableau system TBI with respect to the Grothendieck resource semantics [Galmiche et al. 2005, Theorem 4.3], together with Lemma

new Kripke semantics via soundness and completeness. The PDM semantics are equivalent to the new Kripke semantics [Galmiche et al. 2005, p. 1070f.], and hence the three semantics are equivalent. The reasons for the existence of so many semantics are historical, technical, and also conceptual. The elementary Kripke semantics was the first that was developed, and its simplicity remains conceptually attractive, but BI is not complete with respect to it when the propositional constant $\perp$ is included. Historically, various topological semantics were then developed to solve this problem. The new Kripke semantics solve this problem in a much simpler way, without the topological apparatus. The PDM semantics offer an equivalent, alternative formulation of this solution, and we venture that it represents an important technical and conceptual refinement. Relational semantics seem to have been developed for mainly technical reasons.

Kripke resource semantics, in all of its variants, is a mixture. One ingredient is Kripke's [1965] possible worlds semantics for intuitionistic logic, closely related to his well-known semantics for modal logic [1959, 1963a, 1963b].[7] Kripke's semantics for intuitionistic logic is rather similar to his semantics for the modal logic S4. Both feature an accessibility relation which is reflexive and transitive. The other ingredient is Urquhart's [1972] semantics for relevant logic. Possible worlds are not to be found in Urquhart's account of relevant implication, although they do emerge when he distinguishes relevant implication from a concept of entailment, with the introduction of ideas from Kripke's semantics for modal logic. The account of the meaning of BI's connective $-\!\!*$ is based upon Urquhart's account of relevant implication, and that of $*$ upon his account of intensional conjunction.

2.1. URQUHART'S SEMANTICS FOR RELEVANT LOGICS

Urquhart [1972] gives a semantics for relevant logics. This section presents a summary. Relevant logics are typically characterized as lacking the structural rule of weakening, which allows an argument to be 'watered down' by the addition of arbitrary additional premises, and remain valid. The rule of contraction is typi-

5.6 but which in fact establishes no more than their Theorem 5.5 does, that is, the completeness of TBI with respect to the new Kripke resource semantics.

[7] See Blackburn et al. [2001, §1.3] for a standard, modern account of the semantics of modal logic specified in terms of relational structures.

cally retained. The point that is carried over into the semantics of BI, and which partly expresses the character of BI's multiplicative implication ($-\!\!*$) is that irrelevant information in the antecedent invalidates a relevant conditional: although a proposition may be true given a certain piece of information, it will not be considered true in a relevant logic given the same piece of information taken together with an irrelevant piece of information.

The basic objects of Urquhart's semantics are *pieces of information*, that is, sets of basic sentences of some sort. Pym et al. sometimes seem to proceed as if Urquhart's "pieces of information" were in fact possible worlds. It is a commonly-held intuition that a proposition may be identified, or at least associated with, the set of possible worlds in which it holds. It is also quite reasonable, generally, to think of possible worlds not as alternate universes, but simply as *states* or *points*. Worlds are not typically conceived as divisible objects, and they are not generally thought of as entering into compounds. *Pieces of information*, on the other hand, may in some cases be divisible, and may always enter into compounds. Pieces of information can be combined using set-theoretic union $\cup$, and the empty set $\varnothing$ is regarded as the empty piece of information, or 'no information'. Of course, the $\cup$ operation is idempotent, that is, $X \cup X = X$. This property in particular will be dispensed with when these semantics are adapted to the semantics of BI. In fact, the representation of a piece of information as a set disappears entirely. The property $X \cup X = X$ reflects the availability of the rule of contraction in relevant logics, which is a key point of difference with non-exponential fragment of classical linear logic – that is, multiplicative additive linear logic (MALL). In relevant logics, the number of occurrences of a premise, given that it occurs at all, does not matter: multiple occurrences are logically equivalent to a single occurrence, thanks to contraction. The absence of that rule in MALL, on the other hand, means that the number of occurrences is logically significant.

Relevant logics are typically characterized by a denial of the structural rule of weakening, so that the validity of a valid argument is not in general preserved when additional premises are added to it. The most important feature of the semantics is that it does not in general hold that if $X \Vdash \phi$, then $X \cup Y \Vdash \phi$. A proposition may be rendered false by the availability of additional, irrelevant information – imagine yourself lost in a library. The is an essential ingredient in the meaning BI's connective $-\!\!*$. The intuition behind relevant logic is that every premise, or piece of information, must contribute to a proposition; that it must bear directly upon it in some way. A strong motivation is the resolution of the so-called paradoxes of material (that is, classical) implication, such as $\phi \to (\psi \to \phi)$, which is precisely the formulation of the rule of weakening as an axiom, and $\neg\phi \to (\phi \to \psi)$. The Kripke resource semantics of BI retain the idea of composition of pieces of information, replacing sets of sentences and set-theoretic $\cup$ with a non-set-theoretic view of resources, conceived as the possible worlds of Kripke semantics, together with a composition operation $\bullet$.

$X, Y, \ldots \in S$ are pieces of information, that is, sets of basic sentences. We regard these sentences as semantic objects, not as sentences of the language at hand. S is partially ordered by non-strict set inclusion $\subseteq$, and is a join-semilattice with set-union $\cup$ as the join operation. S always contains the empty set, or empty piece of information $\varnothing$ as its infimum. The $\cup$ operation is commutative, associative, idempotent and has unit $\varnothing$ as would be expected. $\phi, \psi, \ldots$ are propositions, atomic or compound, and $p, q, \ldots \in \mathcal{P}$ are propositional variables. Urquhart states the semantics by way of the specification of a valuation function V, mapping any proposition paired with a piece of information into $\{\mathbf{T}, \mathbf{F}\}$.

We present the semantics using a forcing relation $\Vdash$ (and "does not force", $\nVdash$).[8] We will also use a valuation function V in a different way to Urquhart. Although there is an unfortunate clash of notation, we want to maintain uniformity with the rest of our presentation, Essentially, we use a forcing relation $\Vdash$ instead of Urquhart's $V : \Phi \times S \to \{\mathbf{T}, \mathbf{F}\}$, and we use $V : \mathcal{P} \to \wp(S)$ in our uniform style, with $X \in V(p)$ where Urquhart would write "X determines p".[9] The semantic

[8]We have adopted a uniform notation based upon that used for the semantics of modal logic as presented by Blackburn et al. [2001, §1.3]. Rather than giving the inductive semantic clauses for connectives directly in terms of a model's valuation function, we use forcing notation $\Vdash$ (and $\nVdash$) to indicate that a proposition is made true (or is not made true) in a certain model, usually for a certain state of the model (that is, at a certain possible world, or resource, or for a certain piece of information). For example, $\mathfrak{M}, m \Vdash \phi$ states that ϕ is true at state m in the model $\mathfrak{M}$. We might instead have written something like $V(m, \phi) = \mathbf{T}$, where V is the valuation function of the model $\mathfrak{M}$.

[9]Urquhart's statement of the atomic case runs: "$V(p, X) = \mathbf{T}$ if X determines p, $V(p, X) = \mathbf{F}$ otherwise.", commenting that "A piece of information X may *determine* a basic statement p in the sense that it may be concluded the p is true on the basis of the sentences in X." [Urquhart 1972, p. 160].

treatment of an atomic proposition is really just a place-holder for a philosophical or epistemological specification of the determination of the truth of falsity of a proposition by some information. Note that this formulation is not a strict 'relevantist' one as might be expected; there is no requirement, for instance, that every sentence in X must in some way contribute to the determination of p. This kind of constraint operates only the object language. Determination is much more like a standard valuation function, with a common-sense flavour.

We also introduce the apparatus of frames and models. We define a frame $\mathfrak{F} = \langle S, \emptyset, \cup \rangle$, and a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle = \langle S, \emptyset, \cup, V \rangle$. We write each of the relevant connectives using Urquhart's symbol, subscripted with an r. We have to be careful not to confuse $\rightarrow_r$ with other implicative connectives; and particularly not Urquhart's intensional conjunction $\circ_r$ with the operation of composition of resources $\bullet$ in the semantics of BI. This connective is the ancestor of BI's $*$. Relevant implication $\rightarrow_r$ must also be distinguished from entailment $\rightarrow_e$, which we will touch upon later.

Our formulation of the forcing clause for an atomic proposition, then, is:

$$\mathfrak{M}, X \Vdash p \text{ iff } X \in V(p)$$

An atomic proposition p is forced by the piece of information X iff it is "determined" by X. The forcing clauses for the connectives that we are most interested in are:

$$\mathfrak{M}, X \Vdash \phi \rightarrow_r \psi \text{ iff for all } Y \text{, either } \mathfrak{M}, Y \nVdash \phi \text{ or } \mathfrak{M}, X \cup Y \Vdash \psi$$
$$\mathfrak{M}, X \Vdash \phi \circ_r \psi \text{ iff for some } Y, Z \text{ such that } X = Y \cup Z,$$
$$\mathfrak{M}, Y \Vdash \phi \text{ and } \mathfrak{M}, Z \Vdash \psi$$

We might write the $\circ_r$ clause in a 'pattern-matching' style for clarity:

$$\mathfrak{M}, Y \cup Z \Vdash \phi \circ_r \psi \text{ iff } \mathfrak{M}, Y \Vdash \phi \text{ and } \mathfrak{M}, Z \Vdash \psi$$

The clause for $\rightarrow_r$ says that a relevant conditional holds for a piece of information iff for any piece of information for which the antecedent holds, the consequent holds for the union of the two pieces of information. The clause for $\circ_r$ says that an intensional conjunction holds for a piece of information iff that piece of information is the union of two pieces of information, one for which the left conjunct holds, and the other for which the right conjunct holds. Conjunction and disjunction are as in normal classical and intuitionistic[10] settings; negation is classical and does not figure any further for us.

$$\mathfrak{M}, X \Vdash \phi \mathbin{\&_r} \psi \text{ iff } \mathfrak{M}, X \Vdash \phi \text{ and } \mathfrak{M}, X \Vdash \psi$$
$$\mathfrak{M}, X \Vdash \phi \vee_r \psi \text{ iff } \mathfrak{M}, X \Vdash \phi \text{ or } \mathfrak{M}, X \Vdash \psi$$
$$\mathfrak{M}, X \Vdash \neg\phi \text{ iff } \mathfrak{M}, X \nVdash \phi$$

A formula ϕ is said to be *valid* if for any semilattice S and any valuation function V, we have $\varnothing \Vdash \phi$, that is, if ϕ is determined by the empty piece of information in any circumstances. In the language of frames and models, we define validity this way: A formula ϕ is valid in a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ iff $\mathfrak{M}, \varnothing \Vdash \phi$, abbreviated $\mathfrak{M} \Vdash \phi$; ϕ is valid in a frame $\mathfrak{F}$ iff $\mathfrak{M} \Vdash \phi$ for every valuation V, abbreviated $\mathfrak{F} \Vdash \phi$; and ϕ is *valid* iff $\mathfrak{F} \Vdash \phi$ for every frame $\mathfrak{F}$, that is, for every semilattice S. Although pieces of information are noted to form a semilattice S, the semilattice's partial order does not actually play a prominent role in the semantics, except that the empty, that is, least, piece of information $\varnothing$ is used to define validity. So although there exists a partial order over pieces of information generally, it is important to note that this ordering has little rôle in the semantics for relevant logics, apart from the fact that every semilattice has a common least element.

When the law of idempotency $X \cup X = X$ is suspended, the multiplicative fragment of the relevant logic semantics (that is, the fragment containing just the connectives $\circ_r$ and $\rightarrow_r$) becomes a semantics for multiplicative intuitionistic linear logic (MILL).[11] Linear logic, because contraction is no longer admissible, and weakening never was; intuitionistic because it lacks a *par* connective ($\parr$); and multiplicative because it contains only the remaining multiplicative conjunction and implication.

Urquhart distinguishes the richer notion of *entailment* from relevant implication. For the semantic account of entailment ($\rightarrow_e$) he introduces possible worlds *in addition* to pieces of information, to express the modal content of the notion. Possible worlds are used to represent totalities of facts, which form the background of a judgement as to whether a piece of information determines a proposition. As such, 'quantities' of background facts are not 'weighed and measured' with the

[10] Excepting the absence of intuitionistic monotonicity.
[11] A transition that Pym et al. make without comment.

same requirements of relevance we have for pieces of information. This idea of a factual background resembles Kripke's idea of possible worlds in the semantics of intuitionistic logic as "evidential situations". An entailment is modal because a proposition p must be determined by a piece of information X against all possible factual backgrounds. As in Kripke's semantics for modal logic, the accessibility relation between worlds holds when a world is possible, or accessible from the standpoint of another, that is $m \sqsubseteq n$ iff n is possible from the point of view of m. The possible worlds are preordered by this accessibility relation.[12] We say that $\langle \mathfrak{M}, X, m \rangle \Vdash p$ iff atomic proposition p is determined by the piece of information $X \in S$ against the factual background at the possible world $m \in M$, in the model $\mathfrak{M} = \langle M, \sqsubseteq, S, \varnothing, \cup, V \rangle$. A proposition is valid iff it is determined by the empty piece of information $\varnothing$ against the factual background of every possible world, in every model $\mathfrak{M}$. The forcing clause for entailment is:

$$\begin{aligned}&\mathfrak{M}, X, m \Vdash \phi \rightarrow_e \psi \text{ iff for all } Y, \text{ and all } n \text{ such that } m \sqsubseteq n,\\ &\quad \text{either } \mathfrak{M}, Y, n \nVdash \phi \text{ or } \mathfrak{M}, X \cup Y, n \Vdash \psi\end{aligned}$$

This rule has two domains of semantic objects: pieces of information, for which we have a combining operation $\cup$ and possible worlds, for which we have a preorder relation $\sqsubseteq$. The semantics of BI use a single domain of semantic entities, namely possible worlds or states, or *resources*. These have both a binary combination $\bullet$ and a preorder $\sqsubseteq$ defined over them. The preorder is a preorder of possible worlds, and not a partial order of pieces of information.

2.2. KRIPKE'S SEMANTICS FOR INTUITIONISTIC LOGIC

Kripke introduced the idea of possible worlds for his semantics for modal logic [1959, 1963a, 1963b]. His semantics for intuitionistic logic [1965] is an extension of that project. This section presents a summary of the semantics for intuitionistic logic. Kripke defines *model structures*[13] $\langle G, K, \sqsubseteq \rangle$, in which K is a non-empty

[12] The same holds for the accessibility relation in the semantics of the modal logic S4, as Urquhart notes, but observe that the accessibility relation for S5 is in addition symmetric. See Kripke [1963b] or Hughes & Cresswell [1996] for an overview.

[13] A model structure, but lacking a specially selected actual world, is called a *frame* in modern terminology.

set of possible worlds, $G \in K$ represents the *actual* world, and $\sqsubseteq$ is a reflexive and transitive binary relation between worlds.[14] For the most part, we will omit consideration of G, and just consider model structures, or frames $\mathfrak{F} = \langle K, \sqsubseteq \rangle$. G does, however, figure in the definition of validity. A formula ϕ is *valid* iff $V(\phi, G) = \mathbf{T}$ for every valuation function V over a model structure $\langle G, K, \sqsubseteq \rangle$. That is, ϕ is valid iff it is true at the actual world G under any valuation. G also figures and in Kripke's informal interpretation of the semantics. A *model* is a model structure taken together with a valuation function $V : \Phi \times K \to \{\mathbf{T}, \mathbf{F}\}$ which maps a proposition taken together with a possible world into $\{\mathbf{T}, \mathbf{F}\}$.[15] We write down a model as $\langle K, \sqsubseteq, V \rangle$, and denote it by $\mathfrak{M}$. The valuation function must satisfy the constraint that:

$$\text{If } V(p, m) = \mathbf{T} \text{ and } m \sqsubseteq n \text{ then } V(p, n) = \mathbf{T}$$

This is sometimes called *Kripke monotonicity*. It may be generalised, by induction on formula depth, over all formulæ ϕ for a model $\mathfrak{M}$ in our uniform notation:

$$\text{If } \mathfrak{M}, m \Vdash \phi \text{ and } m \sqsubseteq n \text{ then } \mathfrak{M}, n \Vdash \phi$$

The semantic account of atomic propositions is really a philosophical matter; we take as read that the valuation V for a given model assigns a truth value from $\{\mathbf{T}, \mathbf{F}\}$ to every pair (m, p) of a possible world m and propositional variable p. As far as we are concerned, this assignment may be arbitrary, subject to monotonicity.

$$\mathfrak{M}, m \Vdash p \text{ iff } V(m, p) = \mathbf{T}$$

For convenience, we define:

$$\mathfrak{M}, m \nVdash \phi \text{ iff not } \mathfrak{M}, m \Vdash \phi$$

It is worth noting that something like the law of excluded middle *does* apply at

[14] We write $\sqsubseteq$ for Kripke's R, for uniformity, and to emphasize that it is a preorder. It is most important to note that in many presentations of the semantics of BI [Pym 1999, Pym 2002, O'Hearn & Pym 1999, Pym, O'Hearn & Yang 2004] the direction of the relation is the reverse of ours, that is, where we write $m \sqsubseteq n$, they write $n \sqsubseteq m$. We follow what we take to be the more natural presentation of Kripke [1965], Galmiche & Méry [2001, 2003] and Galmiche, Méry & Pym [2005].

[15] In Kripke's formulation, the valuation function is called the model, and a model is said to be defined over a model structure. In ours, a valuation function is associated with a frame, forming a model.

the level of the valuation function: *every* propositional variable is assigned a truth value at *every* possible world. So this semantics for intuitionistic logic has a classical base. The valuation does not assign truth values for compound propositions. This work is done inductively by the forcing clauses for the connectives:

$\mathfrak{M}, m \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, m \Vdash \phi$ and $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \vee \psi$ iff $\mathfrak{M}, m \Vdash \phi$ or $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \rightarrow \psi$ iff for all n such that $m \sqsubseteq n$, $\mathfrak{M}, n \nVdash \phi$ or $\mathfrak{M}, n \Vdash \psi$
$\mathfrak{M}, m \Vdash \neg\phi$ iff for all n, $m \sqsubseteq n$, $\mathfrak{M}, n \nVdash \phi$

If we add the propositional constant $\perp$, having the forcing clause:

$\mathfrak{M}, m \Vdash \perp$ never

then we can define negation $\neg\phi \equiv \phi \rightarrow \perp$ and drop the clause for negation. Pym et al. opt for this type of presentation of clauses for implication:

$\mathfrak{M}, m \Vdash \phi \rightarrow \psi$ iff for all n such that $m \sqsubseteq n$, $\mathfrak{M}, n \Vdash \phi$ implies $\mathfrak{M}, n \Vdash \psi$

but we prefer to spell out "*A* implies *B*" as "not *A*, or *B*", like Kripke, rather than to hide it. Simply writing "implies" without explication can be a bit obscure, and it also conceals some of the classical reasoning in the model theory.

The general idea for intuitionistic implication is that a conditional obtains iff at any greater or equal possible world – informally, any equal or fuller state of evidence – the consequent is true if the antecedent is true. When we have more evidence, we can prove more, and importantly, things that are true stay true when more evidence is obtained. In intuitionistic terms, of course, it makes better sense to talk about what we can prove, rather than what is true (for a given state of evidence).

The clauses for conjunction and disjunction are classical ones, but the clause for intuitionistic implication is quite different. In this scheme, we would write the clause for classical implication ($\rightarrow_c$) thus:

$\mathfrak{M}, m \Vdash \phi \rightarrow_c \psi$ iff $\mathfrak{M}, m \nVdash \phi$ or $\mathfrak{M}, m \Vdash \psi$

that is, without reference to any other possible world.

It is worth noting that in Kripke's informal interpretation of the semantics for intuitionistic logic, possible worlds are viewed as "evidential situations", that is, situations in which we have access to certain, but not other information. This gives us an interpretation of the preorder over possible worlds quite different to the accessibility relation of one world to another familiar from the Kripke semantics for modal logic. A possible world is greater than or equal, in this information ordering, to another world, if all the information available at the lesser world, and perhaps some more, is available at the greater world. The actual, or least world $G \in K$ in Kripke's model structures marks out the present evidential situation, which forms the root of a tree structure constructed according to the decomposition of the relation R into single, transitive steps S. So we have a ready-made interpretation of intuitionistic logic in terms of the availability of information or evidence. One notable feature of the interpretation is that although we cannot forget information once we have obtained it, we can miss out on future opportunities to obtain certain information because of choices we make now. We may climb the wrong branch of the tree, so to speak.

So we have reasonable semantics for two quite different sorts of logical systems, both of which appeal at some point or other to the semantic work done by bodies or pieces of information. We combine – syntactically and semantically – intuitionistic logic with an adaptation (namely MILL) of the multiplicative fragment of Urquhart relevant logic, to produce BI, which may be understood as a logic of resources.

2.3. Elementary Kripke resource semantics

We now present a standard formulation of the Kripke resource semantics for BI. The principal source is Galmiche, Méry & Pym [2005], but Pym [2002] and Pym, O'Hearn & Yang [2004] are also important sources.

We use the usual language for propositional BI. $p, q, \ldots \in \mathcal{P}$ are proposition letters. $m, n, \ldots \in M$ are *states* of a model, also called possible worlds or resources. $\sqsubseteq$ is a preorder, that is, a reflexive, transitive binary relation on M. $\sqsubseteq$ may be partially defined. That is, not every pair of states need be comparable under $\sqsubseteq$.

Definition 5 A *frame* $\mathfrak{F} = \langle M, \bullet, e, \sqsubseteq \rangle$ is a set of states M and a preorder $\sqsubseteq$, together with a commutative and associative binary operation on states $\bullet : M \times M \to$

M, having the distinguished state $e \in M$ as its unit, such that

$$\text{for all } m \in M, \quad m = m \bullet e = e \bullet m.$$

e is the left and right unit, since $\bullet$ is commutative. A frame is also called a *Kripke resource monoid*. The operation $\bullet$ satisfies a *bifunctoriality* constraint:

$$\text{if } m \sqsubseteq n \text{ and } m' \sqsubseteq n', \text{ then } m \bullet m' \sqsubseteq n \bullet n'$$ □

e is **not** in general the least state of M. $\bullet$ 'combines' two states, or resources, to produce another. It is 'order-preserving' in the sense specified by bifunctoriality. A special case of bifunctoriality is:

$$\text{If } m \sqsubseteq n \text{ then } m \bullet m' \sqsubseteq n \bullet m'$$

which we see in the form of the compatibility constraint for the dependency graphs of resource tableaux. $\bullet$ is not idempotent, that is, in general it is not the case that $m \bullet m = m$, which is the main structural difference between this semantics and Urquhart's: $\bullet$ is the analogue for BI of the operation for combining pieces of information, namely set-theoretic union $\cup$, which *is* idempotent.

It is not in general the case that $m \sqsubseteq m \bullet n$. This property is referred to as *aggregation*. Aggregation may in fact hold in certain classes of frames, and may be useful in modelling certain situations involving resources. Unlike Urquhart's semantics, where pieces of information form a semilattice under the join operation $\cup$, there is no requirement that there be a least element in M. States, or resources, are not required to form a semilattice ordered by the relation $\sqsubseteq$.

Definition 6 A *valuation* function $V : \mathcal{P} \to \wp(M)$ is an assignment of a set of states to each proposition letter. Any assignment must satisfy a *monotonicity* constraint:

$$\text{if } m \in V(p) \text{ and } m \sqsubseteq n, \text{ then } n \in V(p)$$ □

Definition 7 A *model* $\mathfrak{M} = \langle \mathfrak{F}, V \rangle = \langle M, \bullet, e, \sqsubseteq, V \rangle$ is a frame together with a valuation function. □

Given a model, we define a forcing relation $\Vdash$.[16] The forcing clauses are those of Kripke's semantics for $\wedge$, $\vee$ and $\rightarrow$:

$\mathfrak{M}, m \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, m \Vdash \phi$ and $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \vee \psi$ iff $\mathfrak{M}, m \Vdash \phi$ or $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \rightarrow \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$,
  $\mathfrak{M}, n \nVdash \phi$ or $\mathfrak{M}, n \Vdash \psi$

together with clauses for $*$ and $-\!*$ adapted from Urquhart's clauses for $\circ_r$ and $\rightarrow_r$:

$\mathfrak{M}, m \Vdash \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' \sqsubseteq m$,
  $\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \mathrel{-\!*} \psi$ iff for all $n \in M$ such that $\mathfrak{M}, n \Vdash \phi$,
  $\mathfrak{M}, m \bullet n \Vdash \psi$

In addition, BI contains propositional constants $\top$, $\bot$ and I. The clauses for these in the elementary semantics are:

$\mathfrak{M}, m \Vdash \top$ always
$\mathfrak{M}, m \Vdash \bot$ never
$\mathfrak{M}, m \Vdash I$ iff $e \sqsubseteq m$

We say that ϕ is *forced* at a state m in a model $\mathfrak{M}$ if $\mathfrak{M}, m \Vdash \phi$. Since we will need frequently to refer to the forcing clauses, we reproduce them all together in Figure 2.1.

A model is essentially the same as a Kripke model for intuitionistic logic. M is a non-empty set of states (or resources, or possible worlds).[17] The binary relation $\sqsubseteq$ on M is reflexive and transitive, as before. The relation $\sqsubseteq$ in the semantics of BI does not need to be antisymmetric. Antisymmetry means that if $m \sqsubseteq n$ and $n \sqsubseteq m$, then $m = n$. That is, $\sqsubseteq$ need not be a partial order. Indeed, we have no real *formal* requirement for an antisymmetrically derived notion of equality in the semantics, although we are in possession of a notion of equality in virtue of the monoidal structure of frames. Monoidal equality is indeed vital for establishing the equality of various combinations of elements of M under the $\bullet$ operation, but it certainly does not furnish any criterion for establishing identity between distinct primitive

[16] $\mathfrak{M}, m \nVdash \phi$ is just an abbreviation meaning "not $\{\mathfrak{M}, m \Vdash \phi\}$", as before.
[17] We now write the set of states as M instead of Kripke's K.

elements of M. We do however require the ability to recognize the distinguished element $\pi \in M$ which will be introduced with the new Kripke resource semantics in §2.5. But this is not an example of the need for either a 'primitive' or antisymmetric equality that is not derived from the monoid laws: Definition 11 contains the consequence that $m = \pi$ iff $\pi \sqsubseteq m$. There is no stipulation of antisymmetry in Kripke's semantics, nor in his semantics for the modal logic S4, which is similar in essential respects to the intuitionistic semantics. In fact, modal languages generally cannot express antisymmetry in their frames, or rather, equality defined antisymmetrically [Blackburn et al. 2001, §3.3].

Definition 8 (Satisfiability) A formula ϕ is satisfied at a state m in a model $\mathfrak{M}$ if $\mathfrak{M}, m \Vdash \phi$. ϕ is satisfied in a model $\mathfrak{M}$ if for some m, $\mathfrak{M}, m \Vdash \phi$. ϕ is *satisfiable* if it is satisfied in some model $\mathfrak{M}$. □

Definition 9 (Validity)

(i) A formula ϕ is *valid in a model* $\mathfrak{M}$ iff $\mathfrak{M}, m \Vdash \phi$ for every $m \in M$. We write $\mathfrak{M} \Vdash \phi$.

(ii) ϕ is *valid in a frame* $\mathfrak{F}$ iff $\mathfrak{M} \Vdash \phi$ for every valuation V. We write $\mathfrak{F} \Vdash \phi$.

(iii) ϕ is *valid* iff $\mathfrak{F} \Vdash \phi$ for every frame $\mathfrak{F}$. We write $\Vdash \phi$. □

We now propose a weaker notion of validity, which aligns more closely with the notion of theoremhood in BI which we will meet later on.

Definition 10 (e-Validity)

(i) A formula ϕ is e-valid in a model $\mathfrak{M}$ iff $\mathfrak{M}, m \Vdash \phi$ for every $m \in M$ such that $e \sqsubseteq m$.

(ii) ϕ is e-valid in a frame $\mathfrak{F}$ iff ϕ is e-valid in every model $\langle \mathfrak{F}, V \rangle$.

(iii) ϕ is e-valid iff ϕ is e-valid in every frame. □

$\mathfrak{M}, m \Vdash p$ iff $m \in V(p)$
$\mathfrak{M}, m \Vdash \top$ always
$\mathfrak{M}, m \Vdash \bot$ never
$\mathfrak{M}, m \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, m \Vdash \phi$ and $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \vee \psi$ iff $\mathfrak{M}, m \Vdash \phi$ or $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \rightarrow \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$,
  $\mathfrak{M}, n \nVdash \phi$ or $\mathfrak{M}, n \Vdash \psi$
$\mathfrak{M}, m \Vdash I$ iff $e \sqsubseteq m$
$\mathfrak{M}, m \Vdash \phi * \psi$ iff for some $n, n' \in M$ such that $n \bullet n' \sqsubseteq m$,
  $\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi$ iff for all $n \in M$ such that $\mathfrak{M}, n \Vdash \phi$,
  $\mathfrak{M}, m \bullet n \Vdash \psi$

Figure 2.1: The forcing relation for the elementary Kripke resource semantics

2.4. STATES AS RESOURCES

The usual idea for intuitionistic logic is to consider the states of a model not as possible worlds, but as states of knowledge, or *epistemic* states, and the preorder $\sqsubseteq$ as an information ordering on those states. The monotonicity constraint says, effectively, that every ordinary atomic proposition that is known in a given epistemic state is known at any state placed equally or higher in the information ordering. In temporal terms, it says that nothing, once learned, is ever forgotten. With BI, we make the analogy of states with (computational) resources, rather than epistemic states. The preorder $\sqsubseteq$ may then be seen as an ordering on the sufficiency of a resource for some given kind of task. For example, for a task that requires an allocation of memory, 256K $\sqsubseteq$ 512K, in the sense that if a given block of 256K is sufficient, then any block of 512K will be sufficient. Observe that some resources, like computer memory, are fungible, in the economic sense of being a freely interchangeable commodity, and comparable by quantity. Other resources, like URLs, are more reasonably viewed as unique, and are not so easily comparable. Certainly, however, we could specify an information preorder over URLs. There is an obvious sense in which, for example, the URL of the Wikipedia article on XPath `http://en.wikipedia.org/wiki/XPath` lies below the URL of the W3C Recommendation on XPath `http://www.w3.org/TR/xpath` in an information ordering. The Wikipedia article contains less information, and in a certain sense is entirely subsumed by the recommendation, although they are textually distinct; this despite any pedagogical value the Wikipedia article may have, or helpful examples it may contain. Of course a given document might be referenced, or aliased, by more than one URL, or two distinct documents may contain precisely the same information, yielding equivalence in a reasonable information preorder. It is quite natural to collect non-fungible resources into sets and fungible resources into multisets. Hence, for example $\{512\mathrm{K}\} \sqsubseteq \{256\mathrm{K}, 256\mathrm{K}, 256\mathrm{K}\}$ and $\{$`http://en.wikipedia.org/wiki/XPath`$\} \sqsubseteq \{$`http://en.wikipedia.org/wiki/XPath`, `http://en.wikipedia.org/wiki/XQuery`$\}$. In the first example, we suppose some sort of aggregation function on multisets of fungible resources which provides a basis for comparison, and in the second, just upon set inclusion for sets of unique, or non-fungible resources. It would seem reasonable to identify such an aggregation function with the $\bullet$ operation in models. We can see that different sorts of aggregation functions make sense for different sorts of resources. If sets of non-fungible resources are regarded as states of a model, then we can simply define $\bullet$ as set union, and if multisets of fungible resources are the states, we might need to regard $\bullet$ as a more elaborate aggregation of elements of a multiset. Of course, when we have to think of occurrences of the same token in a multiset not as multiple occurrences of the very same object, but as distinct instances of objects of the same type. $\{256\mathrm{K}, 256\mathrm{K}, 256\mathrm{K}\}$ stands for three materially disjoint blocks of memory, and not the same block mentioned three times. We may then want to annotate distinct occurrences of fungible resources, thus: $\{256\mathrm{K}_1, 256\mathrm{K}_2, 256\mathrm{K}_3\}$, and define aggregation such that, for example: $\{256\mathrm{K}_1, 256\mathrm{K}_2, 256\mathrm{K}_3\} \bullet \{256\mathrm{K}_3, 256\mathrm{K}_4\} = \{256\mathrm{K}_1, 256\mathrm{K}_2, 256\mathrm{K}_3, 256\mathrm{K}_4\}$. We may also wish to define aggregation as disjoint, perhaps to avoid computational conflict over resources, so that $\{256\mathrm{K}_1, 256\mathrm{K}_2, 256\mathrm{K}_3\} \bullet \{256\mathrm{K}_3, 256\mathrm{K}_4\}$ might instead be undefined due to the occurrence of $256\mathrm{K}_3$ on both sides. So at least in the fungible case, we may sometimes want a notion of *disjoint* aggregation. We can also imagine situations in which we might want to make use of a disjoint union, for instance to express a constraint that different computational stages or processes use distinct sources of information. This kind of requirement for disjoint combination is accommodated in a straightforward way by the partially-defined monoid semantics which we will come to soon. It simply allows that $\bullet$-combinations of states need not always be defined.[18] More technically, we can envisage restricted variants of BI which correspond to classes of frames in which certain additional properties might hold. For instance, the class of frames in which $\bullet$ and $\sqsubseteq$ satisfy an aggregation property, such that for all m, n it is the case that $m \sqsubseteq m \bullet n$, which is not a general property in standard BI. Or we might develop classes of frames which explicitly identify states with sets of resources or multisets of 'typed' resources, perhaps with disjoint aggregation, and which impose other constraints appropriate to some real-world situation.

2.5. New Kripke resource semantics

This section summarises the new Kripke resource semantics for BI [Galmiche, Méry & Pym 2005, §5.2]. We also prove a standard generalised monotonicity result for this semantics, which holds for all the variations of the semantics presented in this chapter. This new semantics is a small variation on the elementary semantics. The set M of states in a frame must contain an additional distinguished state π, which is the greatest state in the preorder. It is a special state which forces $\perp$ in every model; of course, π is never forced at any other state. The point of including π is essentially to give a representation of $\perp$ that is internal to a model. This semantics is equivalent in semantic strength to the relational and partially-defined monoid semantics that follow. The relational semantics is essentially a different statement of this semantics, and contains the same trick.

Definition 11 A *frame* $\mathfrak{F} = \langle M, \bullet, e, \pi, \sqsubseteq \rangle$ is just the same as for the elementary Kripke resource semantics (Definition 5), except that in every frame, there is another distinguished state $\pi \in M$, called the *inconsistent state*, such that

$$\text{for all } m \in M, \quad m \sqsubseteq \pi \quad \text{and} \quad \pi \bullet m = \pi.$$

Consequently,

$$\pi \sqsubseteq m \text{ iff } m = \pi.$$

[18] An equivalent accommodation can be made in a less obvious way by the new Kripke resource semantics – which will be introduced in the following section – by mapping forbidden combinations of states onto π (see Definition 11).

π is the greatest state in M, under the preorder $\sqsubseteq$, in any frame. $\square$

Definition 12 A *valuation* function $V : \mathcal{P} \to \wp(M)$ is an assignment of a set of states to each proposition letter, as in Definition 6. Any assignment must satisfy the *monotonicity* constraint. By monotonicity, for any $p \in \mathcal{P}$, if there exists $m \in M$ such that $m \in V(p)$, then $\pi \in V(p)$. For reason of uniformity, we further stipulate that for all $p \in \mathcal{P}$, $\pi \in V(p)$, even where p is not satisfied at any other state. $\square$

Definition 13 A *model* $\mathfrak{M} = \langle \mathfrak{F}, V \rangle = \langle M, \bullet, e, \pi, \sqsubseteq, V \rangle$ is a frame together with a valuation function. Given a model, we define a forcing relation $\Vdash$:

$\mathfrak{M}, m \Vdash p$ iff $m \in V(p)$
$\mathfrak{M}, m \Vdash \top$ always
$\mathfrak{M}, m \Vdash \bot$ iff $m = \pi$
$\mathfrak{M}, m \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, m \Vdash \phi$ and $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \vee \psi$ iff $\mathfrak{M}, m \Vdash \phi$ or $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \to \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$,
    $\mathfrak{M}, n \nVdash \phi$ or $\mathfrak{M}, n \Vdash \psi$
$\mathfrak{M}, m \Vdash I$ iff $e \sqsubseteq m$
$\mathfrak{M}, m \Vdash \phi * \psi$ iff for some $n, n' \in M$ such that $n \bullet n' \sqsubseteq m$,
    $\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi$ iff for all $n \in M$ such that $\mathfrak{M}, n \Vdash \phi$,
    $\mathfrak{M}, m \bullet n \Vdash \psi$

$\square$

We require a definition of the depth of a formula for proofs by induction on a formula's complexity. The following routine definition is adapted from Troelstra & Schwichtenberg [2000, p. 10].

Definition 14 The *depth* of a formula is the length of the longest branch in its construction tree. The depth $|\phi|$ of a formula ϕ is defined recursively. Propositional letters and constants have depth 0; and for any binary operator $\circ$, $|\phi \circ \psi| = \max(|\phi|, |\psi|) + 1$. $\square$

The following lemma holds for all the variations of Kripke resource semantics.

Lemma 1 (Generalised Monotonicity)

$$\text{If } \mathfrak{M}, m \Vdash \phi \text{ and } m \sqsubseteq n \text{ then } \mathfrak{M}, n \Vdash \phi$$

□

Proof The proof is by induction on the depth of a formula. We take as the base cases the forcing clauses for propositional letters and constants.

1. In the case that $\mathfrak{M}, m \Vdash p$, we have $m \in V(p)$. If $m \sqsubseteq n$, then by the monotonicity constraint on the valuation of atomic formulæ in Definition 12, we have $n \in V(p)$, and hence $\mathfrak{M}, n \Vdash p$.

2. $\mathfrak{M}, n \Vdash \top$ always, so trivially $\mathfrak{M}, n \Vdash \top$ in the case that $\mathfrak{M}, m \Vdash \top$ and $m \sqsubseteq n$,

3. In the case that $\mathfrak{M}, m \Vdash \bot$, we have $m = \pi$. Then if $m \sqsubseteq n$, then $n = \pi$ by Definition 11, and $\mathfrak{M}, n \Vdash \bot$.

4. In the case that $\mathfrak{M}, m \Vdash I$, we have $e \sqsubseteq m$. Since $m \sqsubseteq n$, we have $e \sqsubseteq n$ by the transitivity of $\sqsubseteq$, and hence $\mathfrak{M}, n \Vdash I$,

In the inductive step, we consider the forcing clause for each binary connective, and show that it preserves monotonicity. In each case, the inductive hypothesis is that if $\mathfrak{M}, m \Vdash \phi$ and $m \sqsubseteq n$, then $\mathfrak{M}, n \Vdash \phi$. In each case, we assume that $m \sqsubseteq n$.

1. Suppose that $\mathfrak{M}, m \Vdash \phi \wedge \psi$. Then $\mathfrak{M}, m \Vdash \phi$ and $\mathfrak{M}, m \Vdash \psi$. By the inductive hypothesis, $\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n \Vdash \psi$, so we have $\mathfrak{M}, n \Vdash \phi \wedge \psi$.

2. Suppose that $\mathfrak{M}, m \Vdash \phi \vee \psi$. Then either $\mathfrak{M}, m \Vdash \phi$ or $\mathfrak{M}, m \Vdash \psi$. By the inductive hypothesis, either $\mathfrak{M}, n \Vdash \phi$ or $\mathfrak{M}, n \Vdash \psi$, so we have $\mathfrak{M}, n \Vdash \phi \vee \psi$.

3. Suppose that $\mathfrak{M}, m \Vdash \phi \to \psi$. Then for all n' such that $m \sqsubseteq n'$, either $\mathfrak{M}, n' \nVdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$. Then since $m \sqsubseteq n$, for all n' such that $n \sqsubseteq n'$, either $\mathfrak{M}, n' \nVdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$, and hence $\mathfrak{M}, n \Vdash \phi \to \psi$ (without use of the inductive hypothesis).

4. Suppose that $\mathfrak{M}, m \Vdash \phi * \psi$. Then there exist n', n'' such that $n' \bullet n'' \sqsubseteq m$, and $\mathfrak{M}, n' \Vdash \phi$ and $\mathfrak{M}, n'' \Vdash \psi$. Since $m \sqsubseteq n$, $n' \bullet n'' \sqsubseteq n$ by the transitivity of $\sqsubseteq$, and so $\mathfrak{M}, n \Vdash \phi * \psi$ (without use of the inductive hypothesis).

5. Suppose that $\mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi$. Then for all n' such that $\mathfrak{M}, n' \Vdash \phi$, we have $\mathfrak{M}, m \bullet n' \Vdash \psi$. By bifunctoriality we have $m \bullet n' \sqsubseteq n \bullet n'$ for all n'. Then for all n' such that $\mathfrak{M}, n' \Vdash \phi$, we have $\mathfrak{M}, n \bullet n' \Vdash \psi$ by the inductive hypothesis, and hence $\mathfrak{M}, n \Vdash \phi \mathrel{-\!\!*} \psi$.

■

Note particularly that cases 3 and 4 of the inductive step do not actually require the inductive hypothesis. It is in this sense that we say that the corresponding forcing clauses have built-in monotonicity. We will return to this point when we make an adjustment to the partially-defined monoid semantics (which might retrospectively be applied in the present case) to build-in monotonicity to the forcing clause for $\mathrel{-\!\!*}$ which is nonetheless conservative for the forcing relation as a whole.

Lemma 2 *$\mathfrak{M}, \pi \Vdash \phi$ for any ϕ in every model $\mathfrak{M}$.* □

PROOF By induction on the degree of a formula. ■

Satisfiability is defined just as for the elementary semantics in Definition 8, except that: ϕ is satisfied in a model $\mathfrak{M}$ if for some m such that $m \neq \pi$, $\mathfrak{M}, m \Vdash \phi$. Validity is defined just as in Definition 9.

2.6. RELATIONAL SEMANTICS FOR BI

This section summarises the relational semantics for BI given by Galmiche, Méry & Pym [2005, §5.1]. This relational semantics is founded upon the insight that we can treat $m \bullet m' \sqsubseteq n$ as a ternary relation $R_{\sqsubseteq}mm'n$.[19]

A frame $\mathfrak{F} = \langle M, e, \pi, R_{\sqsubseteq} \rangle$ contains a ternary relation $R_{\sqsubseteq}$ on states instead of a preorder $\sqsubseteq$ and a composition operator $\bullet$ on states. $e \in M$ is the unit state, and

[19] The compatibility constraint in Figure 2.2, taken together with transitivity, takes the place of the bifunctoriality constraint on $\bullet$. Cf. the compatibility condition on the closure $\overline{K}$ of the set of constraints K on labels for TBI tableaux. The domain function $\mathcal{D}$ gives the set of labels appearing in a set of constraints: if $y \circ z \in \mathcal{D}(\overline{K})$ and $x \leq y \in \overline{K}$ then $x \circ z \leq y \circ z \in \overline{K}$ [Galmiche et al. 2005, §3.1, p. 1045].

Reflexivity	$\forall x \,.\, x \sqsubseteq x$
Commutativity	$\forall x \forall y \forall z \,.\, R_{\sqsubseteq} xyz \rightarrow R_{\sqsubseteq} yxz$
Associativity	$\forall x \forall y \forall z \forall v \,.\, \exists u (R_{\sqsubseteq} xyu \wedge R_{\sqsubseteq} uzv) \leftrightarrow \exists t (R_{\sqsubseteq} yzt \wedge R_{\sqsubseteq} xtv)$
Compatibility	$\forall x \forall y \forall z \forall x' \,.\, R_{\sqsubseteq} xyz \wedge x \sqsubseteq x' \rightarrow R_{\sqsubseteq} x'yz$
Transitivity	$\forall x \forall y \forall z \forall z' \,.\, R_{\sqsubseteq} xyz \wedge z \sqsubseteq z' \rightarrow R_{\sqsubseteq} xyz'$
π-max	$\forall x \forall y \,.\, R_{\sqsubseteq} xy\pi$
π-abs	$\forall x \forall y \,.\, R_{\sqsubseteq} \pi xy \rightarrow \pi \sqsubseteq y$

Figure 2.2: Conditions satisfied by $R_{\sqsubseteq}$

$\pi \in M$ the greatest state, as with the new Kripke resource semantics (Definition 11). For convenience, we define $m \sqsubseteq n \equiv R_{\sqsubseteq} emn$. Any relation $R_{\sqsubseteq}$ must satisfy the conditions given in Figure 2.2.[20] A model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ is a frame taken together with a valuation function V defined just as in Definition 12. Given a model, we define a forcing relation $\Vdash$:

$$
\begin{aligned}
&\mathfrak{M}, m \Vdash p \text{ iff } m \in V(p) \\
&\mathfrak{M}, m \Vdash \top \text{ always} \\
&\mathfrak{M}, m \Vdash \bot \text{ iff } m = \pi \\
&\mathfrak{M}, m \Vdash \phi \wedge \psi \text{ iff } \mathfrak{M}, m \Vdash \phi \text{ and } \mathfrak{M}, m \Vdash \psi \\
&\mathfrak{M}, m \Vdash \phi \vee \psi \text{ iff } \mathfrak{M}, m \Vdash \phi \text{ or } \mathfrak{M}, m \Vdash \psi \\
&\mathfrak{M}, m \Vdash \phi \rightarrow \psi \text{ iff for all } n \in M \text{ such that } m \sqsubseteq n, \\
&\qquad \mathfrak{M}, n \nVdash \phi \text{ or } \mathfrak{M}, n \Vdash \psi \\
&\mathfrak{M}, m \Vdash I \text{ iff } e \sqsubseteq m \\
&\mathfrak{M}, m \Vdash \phi * \psi \text{ iff for some } n, n' \in M \text{ such that } Rnn'm, \\
&\qquad \mathfrak{M}, n \Vdash \phi \text{ and } \mathfrak{M}, n' \Vdash \psi \\
&\mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi \text{ iff for all } n, n' \in M \text{ such that } Rmnn', \\
&\qquad \mathfrak{M}, n \nVdash \phi \text{ or } \mathfrak{M}, n' \Vdash \psi
\end{aligned}
$$

Note that the forcing clause for $\mathrel{-\!\!*}$ in the Kripke resource semantics does not involve $\sqsubseteq$, but that the corresponding clause in this relational semantics does; remember that $Rmnn'$ is equivalent to $m \bullet n \sqsubseteq n'$. It can readily be seen that this

[20]There is a typographical error in the associativity condition in Galmiche et al. [2005, §5.1, p. 1062]. $\forall t$ should read $\forall v$, as here. Otherwise v is free and $\forall t$ does nothing, as occurrences of t are bound by $\exists t$.

formulation strengthens the meaning of multiplicative implication taken in isolation, in comparison with the elementary and new Kripke resource semantics. This modification is conservative with respect to the extension of the forcing relation taken as a whole, but turns out to be crucial to our proof of completeness.[21] The clause builds-in monotonicity in the sense noted earlier. ϕ needs to hold not only at $m \bullet n$ but at any n' such that $m \bullet n \sqsubseteq n'$. That is, it stipulates the monotonicity of ϕ in a way that the elementary and new Kripke resource semantics do not. In the relational semantics, the inductive step case for multiplicative implication in the proof of generalised monotonicity can be proven without appealing to the inductive hypothesis. So although this change strengthens the meaning of the clause for $-\!\!*$ taken alone, it is a conservative modification of Kripke resource semantics.

2.7. PARTIALLY-DEFINED MONOID SEMANTICS FOR BI

This section summarises the partially-defined monoid (PDM) semantics for BI [Galmiche, Méry & Pym 2005, §5.3]. We also make and defend a small modification to this semantics. The PDM semantics presents a different, but equivalent solution to the problem of the completeness of BI with $\perp$, which is handled by the internalisation of inconsistency using π. The shape of this problem will become clearer once we are involved in the details of the completeness proof. The general apparatus is the same as for the elementary Kripke resource semantics, except that that the function $\bullet : M \times M \to M$ may be partially defined. $\downarrow$ is used to indicate that a combination of states is defined: read $m \bullet n \downarrow$ as "$m \bullet n$ is defined".[22] The only constraint on the partial definition of $\bullet$ is associative: that $x \bullet (y \bullet z) \downarrow$ iff $(x \bullet y) \bullet z \downarrow$. The forcing clause for $\perp$ is the same as for the elementary semantics once again, and π does not figure here. The forcing relation is defined as follows:

[21] Galmiche et al. [2005, Theorem 5.2] in fact carry out their completeness proof with respect to this formulation of the semantics, but they do not note this point of difference with the Kripke resource semantics and PDM semantics.

[22] The use of the $\downarrow$ symbol is quite distinct from its proof-theoretic use with the calculus of structures later in the thesis. Although the two uses may come into uncomfortable proximity, no ambiguity will arise. Our use of the symbol is retained in both cases for uniformity with the separate literatures on BI and on the calculus of structures.

$\mathfrak{M}, m \Vdash p$ iff $m \in V(p)$

$\mathfrak{M}, m \Vdash \top$ always

$\mathfrak{M}, m \Vdash \bot$ never

$\mathfrak{M}, m \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, m \Vdash \phi$ and $\mathfrak{M}, m \Vdash \psi$

$\mathfrak{M}, m \Vdash \phi \vee \psi$ iff $\mathfrak{M}, m \Vdash \phi$ or $\mathfrak{M}, m \Vdash \psi$

$\mathfrak{M}, m \Vdash \phi \to \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$,
$\mathfrak{M}, n \nVdash \phi$ or $\mathfrak{M}, n \Vdash \psi$

$\mathfrak{M}, m \Vdash I$ iff $e \sqsubseteq m$

$\mathfrak{M}, m \Vdash \phi * \psi$ iff for some $n, n' \in M$ such that $n \bullet n' \downarrow$ and $n \bullet n' \sqsubseteq m$,
$\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$

$\mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi$ iff for all $n \in M$ such that $m \bullet n \downarrow$ and $\mathfrak{M}, n \Vdash \phi$,
$\mathfrak{M}, m \bullet n \Vdash \psi$

We have to refine the bifunctoriality constraint for the partially defined setting. In the various wholly-defined monoid semantics, every $\bullet$-expression having defined constituents is defined, so any apparent existential import of bifunctoriality is beside the point. The interpretation of bifunctoriality can fall between two extremes. On the one hand, we could adopt a weak interpretation, which makes no guarantees about the definedness of any $\bullet$-expressions:

$$\text{If } m \sqsubseteq n \text{ and } m' \sqsubseteq n' \text{ and } m \bullet m' \downarrow \text{ and } n \bullet n' \downarrow, \text{ then } m \bullet m' \sqsubseteq n \bullet n'.$$

This would mean that no proof could not rely on bifunctoriality to produce existential information about combinations of states, just ordering information about them, should they exist. This would present no difficulty for our completeness proof, but would mean that we could not prove generalised monotonicity for the PDM semantics (cf. case 5 of the inductive step of the proof of Lemma 3 below). A strong construal would guarantee the definedness of combinations on the left- and right-hand sides, thus:

$$\text{If } m \sqsubseteq n \text{ and } m' \sqsubseteq n', \text{then } m \bullet m' \downarrow \text{ and } n \bullet n' \downarrow \text{ and } m \bullet m' \sqsubseteq n \bullet n'$$

It is enough, however, to guarantee the definedness of the combination on the left-hand side when the combination on the right is known to be defined, making no guarantee about the right-hand side. We restate bifunctoriality thus:

$$\text{If } m \sqsubseteq n \text{ and } m' \sqsubseteq n' \text{ and } n \bullet n' \downarrow, \text{ then } m \bullet m' \downarrow \text{ and } m \bullet m' \sqsubseteq n \bullet n'.$$

Lemma 3 (Generalised Monotonicity for the PDM semantics)

$$\text{If } \mathfrak{M}, m \Vdash_{\mathsf{PDM}} \phi \text{ and } m \sqsubseteq n \text{ then } \mathfrak{M}, n \Vdash_{\mathsf{PDM}} \phi$$ □

PROOF The proof is the same as the proof of Lemma 1 expect for cases 4 and 5 of the inductive step, which in this case run as follows:

4. Suppose that $\mathfrak{M}, m \Vdash_{\mathsf{PDM}} \phi * \psi$. Then there exist n', n'' such that $n' \bullet n'' \downarrow$ and $n' \bullet n'' \sqsubseteq m$ and $\mathfrak{M}, n' \Vdash_{\mathsf{PDM}} \phi$ and $\mathfrak{M}, n'' \Vdash_{\mathsf{PDM}} \psi$. Since $m \sqsubseteq n$, $n' \bullet n'' \sqsubseteq n$ by the transitivity of $\sqsubseteq$, and since we already have $n' \bullet n'' \downarrow$, $\mathfrak{M}, n \Vdash_{\mathsf{PDM}} \phi * \psi$.

5. Suppose that $\mathfrak{M}, m \Vdash_{\mathsf{PDM}} \phi \mathrel{-\!\!*} \psi$. Then for all n' such that $m \bullet n' \downarrow$ and $\mathfrak{M}, n' \Vdash_{\mathsf{PDM}} \phi$, we have $\mathfrak{M}, m \bullet n' \Vdash_{\mathsf{PDM}} \psi$. By bifunctoriality we have $m \bullet n' \downarrow$ and $m \bullet n' \sqsubseteq n \bullet n'$ whenever $n \bullet n' \downarrow$. Then for all n' such that $n \bullet n' \downarrow$ and $\mathfrak{M}, n' \Vdash_{\mathsf{PDM}} \phi$, we have $\mathfrak{M}, n \bullet n' \Vdash_{\mathsf{PDM}} \psi$ by the inductive hypothesis, and hence $\mathfrak{M}, n \Vdash_{\mathsf{PDM}} \phi \mathrel{-\!\!*} \psi$.

■

We make one modification to the PDM semantics as given by Galmiche et al. We alter the forcing clause for $\mathrel{-\!\!*}$ as follows:

$$\begin{array}{l} \mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi \text{ iff for all } n \in M \text{ such that } m \bullet n \downarrow \text{ and } \mathfrak{M}, n \Vdash \phi, \\ \quad \text{and all } n' \in M \text{ such that } m \bullet n \sqsubseteq n', \quad \mathfrak{M}, n' \Vdash \psi \end{array}$$

We can give a slightly more pleasant equivalent, flattened formulation:

$$\begin{array}{l} \mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi \text{ iff for all } n, n' \in M \text{ such that } m \bullet n \downarrow \text{ and } \mathfrak{M}, n \Vdash \phi \\ \quad \text{and } m \bullet n \sqsubseteq n', \quad \mathfrak{M}, n' \Vdash \psi \end{array}$$

Any formula $\phi \mathrel{-\!\!*} \psi$ forced at m in the unmodified semantics will in any case be forced at any n such that $m \sqsubseteq n$, in virtue of generalised monotonicity, which holds with or without this modification.

$\mathfrak{M}, m \Vdash p$ iff $m \in V(p)$
$\mathfrak{M}, m \Vdash \top$ always
$\mathfrak{M}, m \Vdash \bot$ never
$\mathfrak{M}, m \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, m \Vdash \phi$ and $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \vee \psi$ iff $\mathfrak{M}, m \Vdash \phi$ or $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \rightarrow \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$,
    $\mathfrak{M}, n \nVdash \phi$ or $\mathfrak{M}, n \Vdash \psi$
$\mathfrak{M}, m \Vdash I$ iff $e \sqsubseteq m$
$\mathfrak{M}, m \Vdash \phi * \psi$ iff for some $n, n' \in M$ such that $n \bullet n' \downarrow$ and $n \bullet n' \sqsubseteq m$,
    $\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi$ iff for all $n, n' \in M$ such that $m \bullet n \downarrow$ and $\mathfrak{M}, n \Vdash \phi$
    and $m \bullet n \sqsubseteq n'$, $\mathfrak{M}, n' \Vdash \psi$

Figure 2.3: The forcing relation for the revised PDM semantics

Proposition 1 *The revised PDM semantics, with the modified forcing clause for $\mathrel{-\!\!*}$, is equivalent to the standard PDM semantics.* □

PROOF We label the forcing relation of the revised PDM semantics $\Vdash_{\mathsf{PDM}'}$. The two semantics differ only in the forcing clause for $\mathrel{-\!\!*}$. Suppose that $\mathfrak{M}, m \Vdash_{\mathsf{PDM}} \phi \mathrel{-\!\!*} \psi$. Then for all $n \in M$ such $\mathfrak{M}, n \Vdash_{\mathsf{PDM}} \phi$ and $m \bullet n \downarrow$, we have $\mathfrak{M}, m \bullet n \Vdash_{\mathsf{PDM}} \psi$. For all $n' \in M$ such that $m \bullet n \sqsubseteq n'$, we have $\mathfrak{M}, n' \Vdash_{\mathsf{PDM}} \psi$ by Lemma 3. Hence if the PDM clause holds, the revised PDM clause holds. Suppose that $\mathfrak{M}, m \Vdash_{\mathsf{PDM}'} \phi \mathrel{-\!\!*} \psi$. Then for all $n, n' \in M$ such that $m \bullet n \downarrow$ and $\mathfrak{M}, n \Vdash_{\mathsf{PDM}'} \phi$ and $m \bullet n \sqsubseteq n'$, we have $\mathfrak{M}, n' \Vdash_{\mathsf{PDM}'} \psi$. Then for all $n \in M$ such that $m \bullet n \downarrow$ and $\mathfrak{M}, n \Vdash_{\mathsf{PDM}'} \phi$ we have $\mathfrak{M}, m \bullet n \Vdash_{\mathsf{PDM}'} \psi$, since $m \bullet n \sqsubseteq m \bullet n$. Hence if the revised PDM clause holds, the PDM clause holds. ■

The motivation for this modification is that it is technically necessary for the success of our completeness proof in §3.4.4. We are also following the precedent that has already been quietly set by the relational semantics. We use a countermodel construction in which states of the countermodel are so-called prime theories; essentially sets of formulæ which are closed under deducibility. Plainly, not every set of formulæ is a prime theory, and we require the preorder over the elements of countermodels to have all non-empty sets of formulæ as its domain, not just the prime theories. We want $m \bullet n$ to be meaningful in the preorder whenever it is defined, even when it is not a first-class state of the model, that is, not a prime theory.

Essentially, we will define $\bullet\colon M \times M \to \wp(\Phi)$ and $\sqsubseteq\colon \wp(\Phi) \times \wp(\Phi) \to \mathsf{Bool}$, with $M \subset \wp(\Phi)$, for the purposes of the completeness proof. We acknowledge the objection that in this case $\langle M, \bullet, e\rangle$ is no longer a monoid. This is a significant fault, introduced only out of technical exigency. We regard the problem of it correction as open.[23]

[23]The most promising approach been unsuccessful. The idea was to find a definition of $\bullet$ for use in the completeness proof to supersede Definition 33 (which also defines $\star$), under which we could define $m \bullet n$ as a unique prime theory when m and n are prime theories. Unfortunately, although $m \star n$ will always have a prime extension, there appears to be no way to define a *unique* least prime extension. We investigated the conjecture that $\bigcap(m \star n)^+$, the intersection of all prime extensions of $m \star n$, which is a sub-prime extension of $m \star n$ (see Definition 32) and is less than or equal to every prime extension of $m \star n$ (see Lemma 20), might always be a prime theory in the case that m and n are prime theories, but we were unable to find a proof. In this case, we could define $\bullet : M \times M \to M$:

$$m \bullet n = \begin{cases} \bigcap(m \star n)^+ & \text{if } m \star n \nvdash \bot \\ \mathsf{undefined} & \text{otherwise} \end{cases}$$

Other unsuccessful investigations have involved significant restructuring of the completeness proof.

Chapter 3

BI in the calculus of structures

In this chapter we present a formulation of the propositional logic of bunched implications (BI) in the calculus of structures. An encoding of sequents into structures is proposed. BI is an asymmetric system, so we propose a straightforward variation of the definition of *polarity* used by Tiu [2005] for intuitionistic logic.

3.1. BI in the calculus of structures, I

For this formulation we draw heavily upon the formulation of the system SJSg for propositional intuitionistic logic given by Tiu [2005].[1] Straßburger's [2003] treatment of multiplicative exponential linear logic (MELL) has been influential. The logic of bunched implications BI, like intuitionistic logic, is asymmetric and lacks involutive negation. BI also has a wider range of connectives than usual, irreducible to one another, and including, of course, the two implications.

Definition 15 A *structure* is defined by the grammar:

$$R ::= a \mid \top \mid \bot \mid I \mid (R;R) \mid [R;R] \mid \langle R;R\rangle \mid (R,R) \mid \langle R,R\rangle$$

Upper-case letters $R, T, U, \ldots$ are structures. We reserve the letter S to denote

[1] Our system SBISg is based upon Tiu's system SJSg in the April 2005 draft of the paper. The down-fragment of SJSg is labelled JSg and the up-fragment cJSg, and we will follow this convention. The LPAR 2006 version of the paper [Tiu 2006] presents a different collection of systems of (quantified) intuitionistic logic in the calculus structures, labelled I instead of J. The g, for "general", indicates the non-local status of the system, that is, that it has non-atomic identity, cut and structural rules.

an entire structure, never a substructure. Lower-case letters $a, b, c, ...$ are atomic propositional structures. $\top$, $\bot$ and I are the propositional constants as structures. $\square$

Square parentheses indicate a disjunctive structure, round parentheses a conjunction, and angled parentheses an implication, as in formulations of intuitionistic logic. We adopt the convention, used for bunches in LBI, that ";" additively connects structures in forming a new structure, and "," multiplicatively connects structures. An expression $S\{R\}$ is called a *context*, and indicates a structure containing a substructure R at arbitrary depth. When the substructure is a tuple construction, the braces may be omitted, e.g. $S[R,T]$. $S\{\ \}$ represents a structure with a *hole* in it. We define equality of structures as shown in Figure 3.1. The unit properties, associativity, commutativity and currying arise directly from isomorphisms in the categorical models, and also reflect proof-theoretic equivalences, i.e. provability both ways in LBI. Congruence is a natural equality for tree structures, and is similar to congruence of bunches. Because of associativity, nested structures of the same sort may be flattened, disregarding nested parentheses. For instance, we may write $[R; [T; [U; V]]]$ as $[R; T; U; V]$. We may also write $\langle (R; T); U\rangle$ as $\langle R; T; U\rangle$, or $\langle (R, T), U\rangle$ as $\langle R, T, U\rangle$. These are just abbreviations, and empty and singleton tuples do not occur. We might have included equivalences for the distributivity theorems $(\phi \wedge (\psi \vee \chi)) \ast\!\!-\!\!\ast ((\phi \wedge \psi) \vee (\phi \wedge \chi))$ and $(\phi * (\psi \vee \chi)) \ast\!\!-\!\!\ast ((\phi * \psi) \vee (\phi * \chi))$, but these will be provable in any case, and have the drawback that the number of occurrences of ϕ varies on either side of each equivalence. The function $\underline{\cdot}_S$ defined in Figure 3.2 recursively specifies the translation from formulæ into structures, and the function $\underline{\cdot}_L$ defined in the same figure defines the translation from structures into formulæ. p stands for a proposition letter and a an atomic structure: there is a one-to-one mapping between proposition letters and atomic structures. The translation of bunches into structures, which we also call $\underline{\cdot}_S$, is an extension of the translation of formulæ into structures, and is given in Figure 3.3. In this case, the formulæ are singleton bunches. Note that the unit bunches $\varnothing_a$ and $\varnothing_m$ are regarded as equivalent to the propositional constants $\top$ and I, and that the translation is forgetful, in that the distinction between formulæ and bunches is lost in the translation. A sequent $\Gamma \Rightarrow \phi$ is translated $\langle \underline{\Gamma}_S, \underline{\phi}_S \rangle$. We treat the $\Rightarrow$ as a multiplicative implication ($-\!\!*$). This is an arbitrary choice, as discussed earlier.

Units	$[R;\bot] = R \quad (R;\top) = R \quad (R,I) = R$ $\langle\top;R\rangle = R \quad \langle I,R\rangle = R$ $[\top;\top] = \top \quad (\bot;\bot) = \bot \quad \langle\bot;\top\rangle = \top \quad \langle\bot;\bot\rangle = \top$
Associativity	$[R;[T;U]] = [[R;T];U] \quad (R;(T;U)) = ((R;T);U)$ $(R,(T,U)) = ((R,T),U)$
Commutativity	$[R;T] = [T;R] \quad (R;T) = (T;R) \quad (R,T) = (T,R)$
Currying	$\langle(R;T);U\rangle = \langle R;\langle T;U\rangle\rangle \quad \langle(R,T),U\rangle = \langle R,\langle T,U\rangle\rangle$
Congruence	if $R = T$ then $S\{R\} = S\{T\}$

Figure 3.1: Syntactic equality of structures

$$
\begin{array}{rcl}
\textit{atomic}\ \underline{p}_S & = & a \\
\underline{I}_S & = & I \\
\underline{\top}_S & = & \top \\
\underline{\bot}_S & = & \bot \\
\underline{\phi * \psi}_S & = & (\underline{\phi}_S , \underline{\psi}_S) \\
\underline{\phi \mathrel{-\!\!*} \psi}_S & = & \langle \underline{\phi}_S , \underline{\psi}_S \rangle \\
\underline{\phi \wedge \psi}_S & = & (\underline{\phi}_S ; \underline{\psi}_S) \\
\underline{\phi \vee \psi}_S & = & [\underline{\phi}_S ; \underline{\psi}_S] \\
\underline{\phi \to \psi}_S & = & \langle \underline{\phi}_S ; \underline{\psi}_S \rangle
\end{array}
\qquad\qquad
\begin{array}{rcl}
\underline{a}_L & = & p \\
\underline{I}_L & = & I \\
\underline{\top}_L & = & \top \\
\underline{\bot}_L & = & \bot \\
\underline{(R,T)}_L & = & \underline{R}_L * \underline{T}_L \\
\underline{\langle R,T\rangle}_L & = & \underline{R}_L \mathrel{-\!\!*} \underline{T}_L \\
\underline{(R;T)}_L & = & \underline{R}_L \wedge \underline{T}_L \\
\underline{[R;T]}_L & = & \underline{R}_L \vee \underline{T}_L \\
\underline{\langle R;T\rangle}_L & = & \underline{R}_L \to \underline{T}_L
\end{array}
$$

Figure 3.2: Translations between formulæ and structures

$$
\begin{array}{rcl}
\textit{atomic}\ \underline{p}_S & = & a \\
\underline{I}_S & = & I \\
\underline{\top}_S & = & \top \\
\underline{\bot}_S & = & \bot \\
\underline{\phi * \psi}_S & = & (\underline{\phi}_S , \underline{\psi}_S) \\
\underline{\phi \mathrel{-\!\!*} \psi}_S & = & \langle \underline{\phi}_S , \underline{\psi}_S \rangle \\
\underline{\phi \wedge \psi}_S & = & (\underline{\phi}_S ; \underline{\psi}_S) \\
\underline{\phi \vee \psi}_S & = & [\underline{\phi}_S ; \underline{\psi}_S] \\
\underline{\phi \to \psi}_S & = & \langle \underline{\phi}_S ; \underline{\psi}_S \rangle
\end{array}
\qquad\qquad
\begin{array}{rcl}
\underline{\varnothing_m}_S & = & I \\
\underline{\varnothing_a}_S & = & \top \\
\underline{\Gamma ; \Delta}_S & = & (\underline{\Gamma}_S ; \underline{\Delta}_S) \\
\underline{\Gamma , \Delta}_S & = & (\underline{\Gamma}_S , \underline{\Delta}_S)
\end{array}
$$

Figure 3.3: The translation from bunches into structures

A derivation in the calculus of structures is a finite chain of structures, with a single premise at the top and the conclusion at the bottom. The structures are linked by downward applications of rules. A proof is a derivation whose premise is $\top$ or I. A proof-search procedure would typically proceed upwards from the conclusion.

Now we need some rules. Rules may involve more than one connective, and express characteristic inferences of the system. The calculus of structures replaces the rules of identity and cut with a dual pair of *interaction* rules, $\mathsf{i}{\downarrow}$ (the identity) and $\mathsf{i}{\uparrow}$ (the cut). In the selection of rules we need to aim not just for soundness and completeness, but also simplicity and a minimum number of rules. If we conjoin premises intuitionistically, we will able to obtain candidate rules by encoding any rule or derivation of LBI. But to obtain corules, we need an adequate notion of polarity.

3.2. POLARITY

Formulations for classical and linear logic depend on an involutive negation $\overline{R}$, to state each corule. In symmetric sequent calculi, polarity exists inasmuch as sequents are[2] divided into antecedent and succedent, with left- and right-handed pairs of rules. But in these cases symmetry and involution give us enough to obtain duality. In an asymmetric system like intuitionistic logic or BI, all that we have is sidedness. For intuitionistic logic, negative polarity corresponds to the left-hand side (antecedent) and positive polarity to the right-hand side (succedent).

We now give a definition of polarity for BI analogous to definition of polarity given by Tiu [2005] for intuitionistic logic in the calculus of structures. This definition differs inasmuch as it is explicitly decompositional – inwards from the outside of the structure.

Definition 16 The *top-level context* $\{\ \}$ is the context $S\{\ \}$ such that $S\{R\} = R$. $\{\ \}$ may also be called the *empty context*. □

Definition 17 (Polarity) Each context in a structure has either positive or negative polarity, but not both. The polarity of a context is defined recursively:

[2] Or *may be* – linear logic is typically right-sided, but just for convenience.

(i) The top-level context $\{\ \}$ is positive;

(ii) If $[S\{\ \};R]$, $[R;S\{\ \}]$, $(S\{\ \};R)$, $(R;S\{\ \})$, $(S\{\ \},R)$, $(R,S\{\ \})$, $\langle R;S\{\ \}\rangle$ or $\langle R,S\{\ \}\rangle$ is positive, $S\{\ \}$ is positive, otherwise it is negative;

(iii) If $\langle S\{\ \};R\rangle$ or $\langle S\{\ \},R\rangle$ is positive, $S\{\ \}$ is negative, otherwise it is positive.□

This definition is not a standard inductive definition, because it starts with its base case (i) at the outside of a structure, and works inwards to determine the polarity of each context in the structure. This is permissible because structures are always finite in size. We expect polarity to be preserved under equality of contexts.[3] Given a structure S, each substructure R of S occupies a hole $S\{\ \}$ in S – that is, a position in the tree structure – which is either positive or negative, according to this definition. The set of holes in S is partitioned into positive and negative contexts. In the base case, the top-level context $\{\ \}$ is occupied by S itself; a structure that does not occupy a hole in another structure is positive.[4] As Brünnler [2004] points out, the system is asymmetric only because the context $\{\ \}$ is positive. Polarity is a property of a context, that is, of a substructure *relative* to the top-level structure containing it, and without reference to the internal structure of the substructure. Polarity must not be considered – at least in the present formulation[5] – as a property of structures, only as a property of contexts. Inspection of the definition reveals that polarity essentially involves the position of substructures in implications. Context expressions $S\{R\}$ are annotated to indicate polarity – or rather, require it – in rule specifications, thus: $S^+\{R\}$ or $S^-\{R\}$. An annotation $S^+\{R\}$ (respectively $S^-\{R\}$) stipulates that the rule at hand is applicable only in cases where the substructure R occupies a positive (respectively negative) context in S. Polarity-checking when looking for rule applications constitutes a form of *non-locality* in the formulation.[6]

---

[3] $S\{\ \} = S'\{\ \}$ iff $\forall R\,.\,S\{R\} = S'\{R\}$.

[4] More precisely, the context $S\{S'\}$ is positive if $S = S'$.

[5] Tiu [2005] gives a local system for intuitionistic logic that assigns polarity labels to structures based upon an initial application of the definition, but then depends on the conservation of polarity under application of the rules: structures never move between positive and negative contexts.

[6] As does the need to check that two formulæ are equal when attempting to apply, e.g. a (non-atomic) rule of contraction.

Once we have an adequate definition of polarity, we can define duality of rules. In symmetric systems, we can define negation over structures ($\overline{R}$), and taking a rule as an implication in the system, obtain the dual of a rule by contraposition of the implication. Something similar occurs with intuitionistic logic, except that instead of negating substructures under contraposition, the polarity restriction on a rule ρ is inverted, thus:

$$\rho\!\downarrow \frac{S^+\{R\}}{S^+\{T\}} \qquad \rho\!\uparrow \frac{S^-\{T\}}{S^-\{R\}}$$

The question arises, whether a more complex notion of polarity might be required to cope with the coexistence of additive and multiplicative implications, or whether the simple 'sidedness' definition will suffice. Is there any any kind of collision of polarities when the two implications are mixed? Should the polarity of $S\{\ \}$ in $\langle\langle S\{\ \}; R\rangle, T\rangle$ simply be positive, or should it instead take some more exotic value? We conjecture at this point that the simple definition is adequate, and this conjecture will be borne out by the soundness and completeness results later on.

3.3. BI IN THE CALCULUS OF STRUCTURES, II

We are now in a position to make an initial proposal of a system of rules for BI.[7] The system of rules directly mirrors the fact that the proofs of BI form a bicartesian DCC (see §1.4). First, the multiplicative *down* fragment:

$$\mathsf{im}\!\downarrow \frac{S^+\{I\}}{S^+\langle R, R\rangle} \qquad \mathsf{scm}\!\downarrow \frac{S^+(\langle R, T\rangle, \langle U, V\rangle)}{S^+\langle (R, U), (T, V)\rangle} \qquad \mathsf{sim}\!\downarrow \frac{S^+(\langle R, T\rangle, \langle U, V\rangle)}{S^+\langle\langle T, U\rangle, \langle R, V\rangle\rangle}$$

The additive *down* fragment:

$$\mathsf{ia}\!\downarrow \frac{S^+\{\top\}}{S^+\langle R; R\rangle} \qquad \mathsf{sca}\!\downarrow \frac{S^+(\langle R; T\rangle; \langle U; V\rangle)}{S^+\langle (R; U); (T; V)\rangle} \qquad \mathsf{sda}\!\downarrow \frac{S^+(\langle R; T\rangle; \langle U; V\rangle)}{S^+\langle [R; U]; [T; V]\rangle} \qquad \mathsf{sia}\!\downarrow \frac{S^+(\langle R; T\rangle; \langle U; V\rangle)}{S^+\langle\langle T; U\rangle; \langle R; V\rangle\rangle}$$

$$\mathsf{cl}\!\downarrow \frac{S^-(R; R)}{S^-\{R\}} \qquad \mathsf{cr}\!\downarrow \frac{S^+[R; R]}{S^+\{R\}} \qquad \mathsf{wl}\!\downarrow \frac{S^-\{\top\}}{S^-\{R\}} \qquad \mathsf{wr}\!\downarrow \frac{S^+\{\bot\}}{S^+\{R\}}$$

[7] Naming of rules follows this scheme: i = interaction; (first letter) c = contraction; w = weakening; s = "switch"; (second letter) c = conjunction; d = disjunction; i = implication; l = left; r = right; (last letter) m = multiplicative; a = additive.

The multiplicative *up* fragment, consisting of corules of the multiplicative *down* fragment:

$$\mathsf{im}\!\uparrow \frac{S^-\langle R, R\rangle}{S^-\{I\}} \qquad \mathsf{scm}\!\uparrow \frac{S^-\langle (R,U),(T,V)\rangle}{S^-(\langle R,T\rangle,\langle U,V\rangle)} \qquad \mathsf{sim}\!\uparrow \frac{S^-\langle\langle T,U\rangle,\langle R,V\rangle\rangle}{S^-(\langle R,T\rangle,\langle U,V\rangle)}$$

Finally, the additive *up* fragment:

$$\mathsf{ia}\!\uparrow \frac{S^-\langle R; R\rangle}{S^-\{\top\}} \qquad \mathsf{sca}\!\uparrow \frac{S^-\langle (R;U);(T;V)\rangle}{S^-(\langle R;T\rangle;\langle U;V\rangle)} \qquad \mathsf{sda}\!\uparrow \frac{S^-\langle [R;U];[T;V]\rangle}{S^-(\langle R;T\rangle;\langle U;V\rangle)} \qquad \mathsf{sia}\!\uparrow \frac{S^-\langle\langle T;U\rangle;\langle R;V\rangle\rangle}{S^-(\langle R;T\rangle;\langle U;V\rangle)}$$

$$\mathsf{cl}\!\uparrow \frac{S^+\{R\}}{S^+(R;R)} \qquad \mathsf{cr}\!\uparrow \frac{S^-\{R\}}{S^-[R;R]} \qquad \mathsf{wl}\!\uparrow \frac{S^+\{R\}}{S^+\{\top\}} \qquad \mathsf{wr}\!\uparrow \frac{S^-\{R\}}{S^-\{\bot\}}$$

This is very similar to the formulation of intuitionistic logic in Tiu [2005], except that there are separate, structurally similar rules for the additive and multiplicative fragments, and that the structural rules are only available in the additive fragment. The arrangement is similar in LBI. We have also stated two pairs of interaction rules. Observe that scm↓ and sim↓ have the same premise, as do sca↓, sda↓ and sia↓. Likewise their duals.

Note particularly that we follow Tiu [2005] in the handling of the contraction rules, to pave the way for atomic contraction rules in a local system. Contraction rules are given in not only a left-, but a right-handed version. The right-handed version corresponds to contraction on the succedent in a multisuccedent sequent calculus for intuitionistic logic, which overturns the restriction on the succedent that characterises traditional sequent calculi for intuitionistic logic.[8] Recall that in Dragalin's system, contraction is admissible, but that the intuitionistic restriction is embodied in the handling of implication. If you try to prove $\Rightarrow p \vee (p \rightarrow \bot)$, $\rightarrow_R$ will force p in the right context to be discarded, so no axiom can be reached. What a proof would need is to keep p aside until it has dealt with $p \rightarrow \bot$, then 'reintroduce' p to the premise of $p \rightarrow \bot$. With deep inference, this is analogous to permitting *interaction* between the results of transformations in distinct contexts. The corresponding restriction in this case is the absence of any rule operating on an (outermost) disjunctive context $S[-;-]$. Manipulations of disjunctive structures are always confined to an implicative context $S\langle[-;-];[-;-]\rangle$, preventing leakage, analogously to Dragalin's system.

[8]Refer to the remarks on pp. 2f. on intuitionistic logic and restricted succedents, and Dragalin's multisuccedent system for intuitionistic logic.

$$\mathsf{im}{\downarrow}\ \frac{S^+\{I\}}{S^+\langle R, R\rangle} \qquad \mathsf{scm}{\downarrow}\ \frac{S^+(\langle R, T\rangle, U)}{S^+\langle R, (T, U)\rangle} \qquad \mathsf{sim}{\downarrow}\ \frac{S^+(R, \langle T, U\rangle)}{S^+\langle\langle R, T\rangle, U\rangle}$$

$$\mathsf{ia}{\downarrow}\ \frac{S^+\{\top\}}{S^+\langle R; R\rangle} \qquad \mathsf{sca}{\downarrow}\ \frac{S^+(\langle R; T\rangle; \langle U; V\rangle)}{S^+\langle (R; U); (T; V)\rangle} \qquad \mathsf{sda}{\downarrow}\ \frac{S^+(\langle R; T\rangle; \langle U; V\rangle)}{S^+\langle [R; U]; [T; V]\rangle} \qquad \mathsf{sia}{\downarrow}\ \frac{S^+(\langle R; T\rangle; \langle U; V\rangle)}{S^+\langle\langle T; U\rangle; \langle R; V\rangle\rangle}$$

$$\mathsf{cl}{\downarrow}\ \frac{S^-(R; R)}{S^-\{R\}} \qquad \mathsf{cr}{\downarrow}\ \frac{S^+[R; R]}{S^+\{R\}} \qquad \mathsf{wl}{\downarrow}\ \frac{S^-\{\top\}}{S^-\{R\}} \qquad \mathsf{wr}{\downarrow}\ \frac{S^+\{\bot\}}{S^+\{R\}}$$

$$\mathsf{im}{\uparrow}\ \frac{S^-\langle R, R\rangle}{S^-\{I\}} \qquad \mathsf{scm}{\uparrow}\ \frac{S^-\langle R, (T, U)\rangle}{S^-(\langle R, T\rangle, U)} \qquad \mathsf{sim}{\uparrow}\ \frac{S^-\langle\langle R, T\rangle, U\rangle}{S^-(R, \langle T, U\rangle)}$$

$$\mathsf{ia}{\uparrow}\ \frac{S^-\langle R; R\rangle}{S^-\{\top\}} \qquad \mathsf{sca}{\uparrow}\ \frac{S^-\langle (R; U); (T; V)\rangle}{S^-(\langle R; T\rangle; \langle U; V\rangle)} \qquad \mathsf{sda}{\uparrow}\ \frac{S^-\langle [R; U]; [T; V]\rangle}{S^-(\langle R; T\rangle; \langle U; V\rangle)} \qquad \mathsf{sia}{\uparrow}\ \frac{S^-\langle\langle T; U\rangle; \langle R; V\rangle\rangle}{S^-(\langle R; T\rangle; \langle U; V\rangle)}$$

$$\mathsf{cl}{\uparrow}\ \frac{S^+\{R\}}{S^+(R; R)} \qquad \mathsf{cr}{\uparrow}\ \frac{S^-\{R\}}{S^-[R; R]} \qquad \mathsf{wl}{\uparrow}\ \frac{S^+\{R\}}{S^+\{\top\}} \qquad \mathsf{wr}{\uparrow}\ \frac{S^-\{R\}}{S^-\{\bot\}}$$

Figure 3.4: The system SBISg

We may, in fact, propose simpler rules for the multiplicative fragment, similar to Brünnler's [2004] proposal for minimal intuitionistic logic, because we are not constrained by the complications with contraction. We designate the system having these simpler multiplicative rules SBISg, and we present it as a whole in Figure 3.4. The multiplicative fragment, however, no longer has a direct morphological similarity to the 'logical' part of the additive fragment.

Definition 18 We make the following classification of the rules of SBISg:

interaction rules	im↓ ia↓ im↑ ia↑
structural rules	cl↓ cr↓ wl↓ wr↓ cl↑ cr↑ wl↑ wr↑
switch rules	scm↓ sim↓ sca↓ sda↓ sia↓ scm↑ sim↑ sca↑ sda↑ sia↑

Switch rules could also be called *logical* rules. □

3.4. SOUNDNESS AND COMPLETENESS

We now present soundness and completeness results for the system SBISg for propositional BI (with $\bot$) in the calculus of structures. The soundness result is with respect to the elementary Kripke resource semantics. Almost the same argument shows soundness with respect to the new Kripke resource semantics for

BI, featuring the greatest state π. This argument differs only the case for wr↓, as noted below. A slightly more complicated variation on the argument can be used to show soundness with respect to the partially-defined monoid (PDM) semantics for BI; the fairly straightforward complications surrounding the definedness of $\bullet$-expressions in the forcing clauses for $*$ and $-\!\!*$. We can regard soundness with respect to the elementary Kripke resource semantics as the stronger result. The completeness result is for SBISg (with $\bot$) with respect to the PDM semantics for BI.[9] Note that the reasoning in our proofs is classical, not intuitionistic, although we a dealing with a logic of an intuitionistic character. For reference, the rules of SBISg are laid out all together in Figure 3.4, and also earlier on the syntactic equality of structures in Figure 3.1, and the translations between formulæ and structures in Figure 3.2. A structure R is semantically valid iff $\Vdash \underline{R}_L$. We simply write $\Vdash R$. Pym [2002, §6.3] proves soundness and completeness of the propositional sequent system LBI by equivalence (for provability) with the natural deduction system NBI; the soundness and completeness of NBI are proven semantically. Tiu [2005] similarly proves soundness and completeness of the system SJSg of intuitionistic logic in the calculus of structures by proving the equivalence (for provability) of SJSg with the sequent system LJ. We would expect to be able to prove soundness and completeness of SBISg with respect to LBI in a similar way, but we have decided to carry out the proof with respect to a variant of Kripke resource semantics for the insight it provides into the proof theory and semantics of BI.

3.4.1. Proof-theoretical preliminaries

Definition 19 If $\Gamma = \{R_1, \ldots, R_n\}$ is a finite set of structures with two or more elements, then we write $\bigwedge \Gamma$ to denote the structure $(R_1; \ldots; R_n)$, which is the additive conjuction of the elements of Γ. If $\Gamma = \{R\}$, then $\bigwedge \Gamma$ denotes the structure R. If $\Gamma = \varnothing$, then $\bigwedge \Gamma$ denotes the structure $\top$. Γ must be of finite size, since a structure must be of finite size. □

[9] SBISg without $\bot$ is complete with respect to the elementary Kripke resource semantics, but SBISg (with $\bot$) is incomplete with respect to the elementary Kripke resource semantics.

Definition 20 Each rule of inference of SBISg has a dual rule in SBISg. The dual of a rule exchanges the position of premise and conclusion, and inverts the polarity restriction on rule application. Each rule $\rho\downarrow$ has a dual $\rho\uparrow$, and each $\rho\uparrow$ a dual $\rho\downarrow$, where ρ is the string of letters in each rule label. □

Definition 21 A *derivation* of T from R is a finite chain of inferences, downwards, from R to T. We write $\begin{smallmatrix} R \\ \| \\ T \end{smallmatrix}$ to denote such a derivation. We write $R \vdash_{\mathsf{X}} T$ to indicate that there exists a derivation of T from R in a system X. In particular, we write $R \vdash_{\mathsf{SBISg}} T$ to indicate that there exists a derivation of T from R in the system SBISg. We generally omit the subscript and write $R \vdash T$ unless some ambiguity would arise. □

Definition 22 A *proof* of R in a BI-system X is a derivation in X of R from either I or $\top$. We write $\vdash_{\mathsf{X}} R$ to indicate that there exists a proof of the structure R in X. R is a *theorem* of X iff $\vdash_{\mathsf{X}} R$. Accordingly, R is a theorem of SBISg iff there exists a proof of R in SBISg. We write $\vdash_{\mathsf{SBISg}} R$ to indicate that the structure R is provable in SBISg, that is, that R is a theorem of SBISg. We generally write $\vdash R$ unless some ambiguity would arise. □

Theorem 1 *R is a theorem of SBISg iff there is a derivation of R from I.* □

PROOF Since there is a derivation

$$\equiv \frac{\mathsf{wl}\downarrow \dfrac{\mathsf{im}\downarrow \dfrac{I}{\langle \top, \top \rangle}}{\langle I, \top \rangle}}{\top}$$

of $\top$ from I, it is sufficient that there be a derivation of R from I for R to be a theorem, since any derivation of R from $\top$ may be extended upwards to produce a derivation of R from I. If R is derivable from $\top$, then it is derivable from I, but not vice versa. ■

Remark 1 Note the use of the weakening rule $\mathsf{wl}\downarrow$; this move is not available if we try to derive I from $\top$, and there is indeed no derivation of I from $\top$. □

Definition 23 A structure R is *deducible* from a set of structures Γ in a BI-system X if there exists a finite subset Γ' of Γ such that $\bigwedge \Gamma' \vdash_{\mathsf{X}} R$. We generally write $\Gamma \vdash_{\mathsf{X}} R$, to indicate that R is deducible from Γ in X, even though Γ is a set of structures rather than an individual structure. We write $\Gamma \vdash_{\mathsf{SBISg}} R$ to to indicate that R is deducible from Γ in SBISg, and we write $\Gamma \vdash R$ when it is clear which system we are referring to. It is trivial that $\{R\} \vdash_{\mathsf{SBISg}} T$ iff $R \vdash_{\mathsf{SBISg}} T$. $\square$

Recall the isomorphisms of hom-sets

$$[I, A \mathrel{-\!\!*} B] \cong [A, B] \cong [\top, A \to B]$$

that occurs in closed categories in the categorical "semantics" of proofs in BI. These isomorphisms are reflected in SBISg, as illustrated by the following result.

Lemma 4 $\vdash \langle R, T\rangle$ *iff* $R \vdash T$ *iff* $\vdash \langle R; T\rangle$. $\square$

PROOF Given a derivation $\begin{array}{c} R \\ \| \\ T \end{array}$ we can construct derivations $\begin{array}{c} I \\ \| \\ \langle R, T\rangle \end{array}$ and $\begin{array}{c} \top \\ \| \\ \langle R; T\rangle \end{array}$ as follows:

$$\begin{array}{ll} \mathsf{im}\!\downarrow & \dfrac{I}{\langle R, R\rangle} \\ & \dfrac{\Downarrow}{\langle R, T\rangle}\ \text{Paste}\ \begin{array}{c} R \\ \| \\ T \end{array} \end{array} \qquad\qquad \begin{array}{ll} \mathsf{ia}\!\downarrow & \dfrac{\top}{\langle R; R\rangle} \\ & \dfrac{\Downarrow}{\langle R; T\rangle}\ \text{Paste}\ \begin{array}{c} R \\ \| \\ T \end{array} \end{array}$$

If $\begin{array}{c} R \\ \| \\ T \end{array}$ is cut-free, that is, uses only the down-fragment, then so are these derivations.

Of course, each of these derivations is a proof. Given a derivation $\begin{array}{c} I \\ \| \\ \langle R, T\rangle \end{array}$ or a derivation $\begin{array}{c} \top \\ \| \\ \langle R; T\rangle \end{array}$, we can construct a derivation $\begin{array}{c} R \\ \| \\ T \end{array}$, in each case using a cut rule:

$$\begin{array}{ll} \equiv & \dfrac{R}{(R, I)} \\ & \dfrac{\Downarrow}{(R, \langle R, T\rangle)}\ \text{Paste}\ \begin{array}{c} I \\ \| \\ \langle R, T\rangle \end{array} \\ \mathsf{sim}\!\downarrow & \dfrac{}{\langle\langle R, R\rangle, T\rangle} \\ \mathsf{im}\!\uparrow & \dfrac{}{\langle I, T\rangle} \\ \equiv & \dfrac{}{T} \end{array} \qquad\qquad \begin{array}{ll} \equiv & \dfrac{R}{(\langle \top; R\rangle; \top)} \\ & \dfrac{\Downarrow}{(\langle \top; R\rangle; \langle R; T\rangle)}\ \text{Paste}\ \begin{array}{c} \top \\ \| \\ \langle R; T\rangle \end{array} \\ \mathsf{sia}\!\downarrow & \dfrac{}{\langle\langle R; R\rangle; \langle \top; T\rangle\rangle} \\ \mathsf{ia}\!\uparrow & \dfrac{}{\langle \top; \langle \top; T\rangle\rangle} \\ \equiv & \dfrac{}{T} \end{array}$$

$$\text{im}\!\downarrow \frac{S^+\{I\}}{S^+\langle R,R\rangle} \quad \text{scm}\!\downarrow \frac{S^+(\langle R,T\rangle,U)}{S^+\langle R,(T,U)\rangle} \quad \text{sim}\!\downarrow \frac{S^+(R,\langle T,U\rangle)}{S^+\langle\langle R,T\rangle,U\rangle}$$

$$\text{ia}\!\downarrow \frac{S^+\{\top\}}{S^+\langle R;R\rangle} \quad \text{sca}\!\downarrow \frac{S^+(\langle R;T\rangle;\langle U;V\rangle)}{S^+\langle (R;U);(T;V)\rangle} \quad \text{sda}\!\downarrow \frac{S^+(\langle R;T\rangle;\langle U;V\rangle)}{S^+\langle [R;U];[T;V]\rangle} \quad \text{sia}\!\downarrow \frac{S^+(\langle R;T\rangle;\langle U;V\rangle)}{S^+\langle\langle T;U\rangle;\langle R;V\rangle\rangle}$$

$$\text{cl}\!\downarrow \frac{S^-(R;R)}{S^-\{R\}} \quad \text{cr}\!\downarrow \frac{S^+[R;R]}{S^+\{R\}} \quad \text{wl}\!\downarrow \frac{S^-\{\top\}}{S^-\{R\}} \quad \text{wr}\!\downarrow \frac{S^+\{\bot\}}{S^+\{R\}}$$

Figure 3.5: The system BISg, which is the ‘cut-free’ down-fragment of SBISg

$$\text{im}\!\uparrow \frac{S^-\langle R,R\rangle}{S^-\{I\}} \quad \text{scm}\!\uparrow \frac{S^-\langle R,(T,U)\rangle}{S^-(\langle R,T\rangle,U)} \quad \text{sim}\!\uparrow \frac{S^-\langle\langle R,T\rangle,U\rangle}{S^-(R,\langle T,U\rangle)}$$

$$\text{ia}\!\uparrow \frac{S^-\langle R;R\rangle}{S^-\{\top\}} \quad \text{sca}\!\uparrow \frac{S^-\langle (R;U);(T;V)\rangle}{S^-(\langle R;T\rangle;\langle U;V\rangle)} \quad \text{sda}\!\uparrow \frac{S^-\langle [R;U];[T;V]\rangle}{S^-(\langle R;T\rangle;\langle U;V\rangle)} \quad \text{sia}\!\uparrow \frac{S^-\langle\langle T;U\rangle;\langle R;V\rangle\rangle}{S^-(\langle R;T\rangle;\langle U;V\rangle)}$$

$$\text{cl}\!\uparrow \frac{S^+\{R\}}{S^+(R;R)} \quad \text{cr}\!\uparrow \frac{S^-\{R\}}{S^-[R;R]} \quad \text{wl}\!\uparrow \frac{S^+\{R\}}{S^+\{\top\}} \quad \text{wr}\!\uparrow \frac{S^-\{R\}}{S^-\{\bot\}}$$

Figure 3.6: The system cBISg, which is the up-fragment of SBISg

■

It is crucial to note, for later consideration, that this *general* result depends on the application of cut rules, so we cannot rely upon it when we come to consider the ‘cut-free’ down-fragment BISg[10] of SBISg on its own, in relation to the question of cut-elimination for SBISg. This illustrates the fact that a cut-elimination theorem guarantees a cut-free proof for every theorem, but not a cut-free derivation of ψ from ϕ whenever $\phi \vdash \psi$. Generally speaking, we need to use an up-rule in a derivation to make a substructure disappear in the course of a derivation.[11]

Lemma 5 *If* $R \vdash_{\mathsf{BISg}} T$ *then* $\vdash_{\mathsf{BISg}} \langle R,T\rangle$ *and* $\vdash_{\mathsf{BISg}} \langle R;T\rangle$. □

[10]See Figure 3.5. See also Figure 3.6 for the system cBISg, which is the up-fragment of SBISg, and a ‘mirror-image’ of BISg, in that each rule inverts the direction of inference, and flips the polarity restriction, of its counterpart in BISg.

[11]There is a similar point to be make with sequent calculi. In LBI, we have certain proof-theoretical ‘equivalences’ between sequents, not themselves representing theorems, for instance, $\phi,\psi \Rightarrow \chi$ iff $\psi * \psi \Rightarrow \chi$. But to establish such a fact, we need to produce derivations in each direction, and we need to use an application of cut in this instance, in the right-to-left case. Here is the example, and two others, set out in LBI. These are all cases in which the effect of a (single-premise) right-side rule is inverted, in a way reminiscent of the invertibility of rules of inference in systems of natural deduction [see, for the classic treatment, Prawitz 1965], and we cannot carry out these inversions in the cut-free system.

Proof Refer to the proof of Lemma 4 above. ■

A derivation or proof occurs in the top-level context, which has positive polarity. When we attempt to write down a proof, a known derivation may be pasted into a positive context. For example, suppose that we already have a derivation $\begin{array}{c} R \\ \Vert \\ T \end{array}$, and we reach a structure in the attempted proof with R as a substructure in a positive context. Then we can copy the chain of inferences of $\begin{array}{c} R \\ \Vert \\ T \end{array}$ to obtain a structure in which T replaces R in that positive context. Any chain of inferences in a positive context can be 'pasted' into another positive context in the construction of some other proof. Each up-rule is the exact inversion of the corresponding down-rule, with the polarity restriction inverted. This is a manifestation of the up-down symmetry that is characteristic of systems in the calculus of structures.

Definition 24 A *subderivation* of a proof is a chain of one or more applications of rules of inference in the same context. A subderivation in a positive context is called a *positive subderivation*, and a subderivation in a negative context is called a *negative subderivation*. Of course rule applications at greater depth than the top level of a given context may vary in polarity according to the usual polarity rules. Every derivation is a positive subderivation, and every positive subderivation can be plucked out of its context to produce a derivation. □

$$\dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \phi}\,id \qquad \dfrac{}{\psi \Rightarrow \psi}\,id}{\phi, \psi \Rightarrow \phi * \psi}\,*_R \qquad \dfrac{\vdots}{\phi * \psi \Rightarrow \chi}}{\phi, \psi \Rightarrow \chi}\,cut \qquad \text{inverts} \qquad \dfrac{\dfrac{\vdots}{\phi, \psi \Rightarrow \chi}}{\phi * \psi \Rightarrow \chi}\,*_L$$

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow \phi -\!\!* \psi} \qquad \dfrac{\dfrac{}{\phi \Rightarrow \phi}\,id \qquad \dfrac{}{\psi \Rightarrow \psi}\,id}{\phi, \phi -\!\!* \psi \Rightarrow \psi}\,-\!\!*_L}{\Gamma, \phi \Rightarrow \psi}\,cut \qquad \text{inverts the rule} \qquad \dfrac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi -\!\!* \psi}\,-\!\!*_R$$

$$\dfrac{\dfrac{}{\varnothing_a \Rightarrow \top}\,\top_R \qquad \dfrac{\vdots}{\top \Rightarrow \phi}}{\varnothing_a \Rightarrow \phi}\,cut \qquad \text{inverts} \qquad \dfrac{\dfrac{\vdots}{\varnothing_a \Rightarrow \phi}}{\top \Rightarrow \phi}\,\top_L$$

Proposition 2 *Given positive subderivation of T from R (or simply a derivation) we can construct a* negative derivation *of R from T by inverting the entire derivation, simultaneously relabelling each $\uparrow$ rule application with its $\downarrow$ counterpart, and each $\downarrow$ rule application with its $\uparrow$ counterpart. A negative derivation is sound in a negative context: it can be pasted into a negative context in the construction of a proof.* □

Remark 2 Generally speaking, a positive or negative derivation can be pasted into a context of matching polarity, or its inverse derivation pasted into a context of inverse polarity. Of course, proofs are a special case of derivations. Subderivations may be regarded as reusable modules for proof construction. A positive subderivation may be 'plugged-in' to a positive context containing the top-most structure of the subderivation to obtain a structure where the bottom-most structure of the subderivation fills that context. Likewise for a negative subderivation and a negative context. □

The following theorem is a generalisation of Tiu's [2005] Proposition 2.

Lemma 6 (Dual derivations) *Given a positive subderivation $\begin{array}{c} R \\ \| \\ T \end{array}$, we can construct a dual, negative subderivation $-\begin{array}{c} T \\ \| \\ R \end{array}$. Similarly, given a negative subderivation $-\begin{array}{c} T \\ \| \\ R \end{array}$, we can construct a dual, positive subderivation $\begin{array}{c} R \\ \| \\ T \end{array}$.* □

PROOF Construct the dual subderivation, in each case, by inverting the subderivation, and simultaneously relabelling each rule application with the label of its dual. The newly constructed subderivation can be shown to be a correct subderivation in a context of opposite polarity by induction on the length of the derivation and the definitions of the rules of SBISg in Figure 3.4. ■

Remark 3 Of course a dual subderivation can never replace the original subderivation in its original context. It may, however, be used as a module in the construction of new proofs, in the sense of Proposition 2. □

The following 'cut-and-paste' lemma justifies the modular construction of new derivations from known ones.

Lemma 7 (Cut-and-paste lemma) *If there exists a derivation* $\begin{array}{c} S^{+}\{R\} \\ \Vert \\ S^{+}\{T\} \end{array}$ *(in which the context* $S^{+}\{\ \}$ *is a positive context) then for any positive context* $X^{+}\{\ \}$*, there exists a derivation* $\begin{array}{c} X\{R\} \\ \Vert \\ X\{T\} \end{array}$*, and for any negative context* $X^{-}\{\ \}$ *there exists a derivation* $\begin{array}{c} X\{T\} \\ \Vert \\ X\{R\} \end{array}$*. If there exists a derivation* $\begin{array}{c} S^{-}\{R\} \\ \Vert \\ S^{-}\{T\} \end{array}$ *then for any positive context* $X^{+}\{\ \}$*, there exists a derivation* $\begin{array}{c} X\{T\} \\ \Vert \\ X\{R\} \end{array}$*, and for any negative context* $X^{-}\{\ \}$ *there exists a derivation* $\begin{array}{c} X\{R\} \\ \Vert \\ X\{T\} \end{array}$*.* □

Proof First, we can reduce the number of cases by invoking Lemma 6. Then proof is by induction on the length of the derivation. ■

3.4.2. Soundness

To show that SBISg is sound, we might at first expect that we would need to show that each theorem of SBISg is valid in the elementary Kripke resource semantics. We have a peculiar difficulty, however. I is a theorem of SBISg (as well as of LBI and other proof systems for BI) but I is not valid in every model, since e need not be the least state in every model. But a structure R is a theorem of SBISg iff there exists a derivation of R from I or from $\top$, and since there is a derivation of $\top$ from I, a structure R is a theorem of SBISg iff there exists a derivation of R from I. There is a sense in which we are taking for granted – or rather taking on trust from the proof theory – that theorems are those structures that are derivable from I. BI forces us to forego the standard idea that *theoremhood = validity in all models*. Some theorems, like I, are not valid in all models.

Before proceeding further with our preamble, we will give a definition of semantic entailment.

Definition 25 (Semantic entailment) We say that $R \vDash T$ iff in every model $\mathfrak{M}$, for any $m \in M$, either $\mathfrak{M}, m \nVdash R$ or $\mathfrak{M}, m \Vdash T$. That is, $R \vDash T$ iff for any model, if R is forced at any state in that model, then T is forced at that state. □

Remark 4 If follows from $R \vDash T$ that if R is valid in a model, then T is valid in that model, and that if R is valid, then T is valid. Similarly, if R is e-valid in a model, then T is e-valid in that model, and that if R is e-valid, then T is e-valid. □

Consider a comparison with LBI. ϕ is a theorem of LBI iff there is a proof of either the sequent $\varnothing_a \Rightarrow \phi$ or the sequent $\varnothing_m \Rightarrow \phi$. Since there is a derivation of $\varnothing_m \Rightarrow \phi$ from $\varnothing_a \Rightarrow \phi$, it is sufficient that $\varnothing_m \Rightarrow \phi$. Usually, the case in which every theorem is semantically valid is the special case in which the antecedent Γ – to take the example of intuitionistic logic – is the empty multiset, which is conventionally equivalent to $\top$, since $\top$ is forced everywhere. For LBI, there are two such special cases, where Γ is either one of the unit bunches $\varnothing_a$ and $\varnothing_m$. The cases where $\varnothing_m \vdash \phi$ but $\varnothing_a \nvdash \phi$ are precisely those cases where ϕ is a theorem, but is not semantically valid. ϕ is e-valid, however.[12] Consequently, to get around this difficulty, Pym's soundness result for LBI[13] is couched in terms of whole sequents and semantic entailment, rather than in terms of individual formulæ or theorems, and validity. Hence soundness of LBI means that if the sequent $\Gamma \Rightarrow \phi$ is provable, then $\Gamma \vDash \phi$. If we let γ be the formula obtained from the bunch Γ by replacing each "," with "$*$" and each ";" with "$\wedge$", then $\Gamma \vDash \phi$ means that in any model, ϕ is forced at any state at which γ is forced. This form of soundness is a fine-grained result about relative forcing, not about absolute validity, or even relative validity, and it is a strong type of soundness result. The relative form of soundness

[12]It has been suggested that a different notion of theoremhood be considered: that ϕ be a theorem iff $\top \vdash \phi$. In this case, we ought to be able to discard the notion of e-validity. It is suggested that accordingly the rule im↓ would need to be changed, and that only a derivation from $\top$ be counted as a proof. The rule im↑ would have to change as well.

$$\mathsf{im}{\downarrow}\ \frac{S^+\{\top\}}{S^+\langle R, R\rangle} \qquad \mathsf{im}{\uparrow}\ \frac{S^-\langle R, R\rangle}{S^-\{\top\}}$$

It is suggested that in this case, I would no longer be a theorem. But due to the syntactic equality $\langle I, R\rangle = R$, it still would be:

$$\equiv \frac{\mathsf{im}{\downarrow}\ \dfrac{\top}{\langle I, I\rangle}}{I}$$

But $\top \nvDash I$, so the modified rules would make the proof system unsound. We would in any case resist the idea for the sake of uniformity with the literature on BI.

[13]Actually, his soundness result for NBI [Pym 2002, §4.2]. The result for LBI follows from equivalence with NBI [Pym 2002, §6.3].

is stronger inasmuch as it says that for any antecedent that is forced at some state of any model (but not necessarily at all states) the consequent is forced at that state. We can obtain some weaker soundness results as corollaries. For instance, that every theorem of BI is e-valid. Soundness results come in several grades of strength, then. The weakest states that a theorem is valid, or for BI, e-valid; in between we might have a result that if $R \vdash T$, then when R is valid (or e-valid) in an arbitrary model, T is valid (or e-valid) in that model; the strongest is a result at the level of forcing. We will prove a soundness result of the strong form: every inference of SBISg preserves forcing at an arbitrary state of an arbitrary model. (In fact, the special case that if $\vdash R$ then $I \models R$, which treats just the theorems of BI, would suffice as a soundness result of 'middle' strength.) That is, if R is forced at an arbitrary state m in an arbitrary model $\mathfrak{M}$, and there exists a derivation $\begin{array}{c} R \\ \| \\ T \end{array}$, then T is forced at m in $\mathfrak{M}$.

We also have to deal with the complication that inference may be deep inference, that is inference operating not at the top level of a structure, but operating directly on a substructure. Moreover, deep inference may occur in polarised contexts.

Lemma 8 *The rules for syntactic equality of structures preserve the polarity of substructures.* □

Proof By induction on the depth of structures. ■

Remark 5 Depth is not conserved under syntactic equivalence. □

Definition 26 A *top-level derivation* is a derivation of one entire structure from another. A top-level derivation is a positive derivation. □

Definition 27 We say that a rule of inference is *valid* if it preserves (semantic) validity. That is, if the rule is applied with any (semantically) valid premise, the conclusion will also be (semantically) valid. □

Definition 28 A subcontext $S'\{\ \}$ of $S\{\ \}$ is said to be an *immediate subcontext* of $S\{\ \}$ if $S'\{\ \}$ occurs at a structural depth of one in $S\{\ \}$, and (of course) the hole

$\{\ \}$ in each of $S\{\ \}$ and $S'\{\ \}$ is the same hole. For example, $S'\{\ \}$ is an immediate subcontext of $S\{\ \}$ if $S\{\ \} = (R; S'\{\ \})$, but not if $S\{\ \} = \langle R; (T; S'\{\ \})\rangle$ or if $S'\{\ \} = S\{\ \}$. □

We now state and prove a lemma which says essentially that any inference that is valid at the top level of a structure is valid for deep inference in a context of appropriate polarity.

Lemma 9 (Semantic depth) *If $R \models T$ then for any positive context $S\{\ \}$ we have $S\{R\} \models S\{T\}$, and for any negative context $S\{\ \}$ we have $S\{T\} \models S\{R\}$.* □

PROOF We proceed by a structural induction on depth. In this proof $S\{\ \}$ is always a top-level context, and hence always occurs *in* a positive context. $S'\{\ \}$ is always an immediate subcontext of $S\{\ \}$. Note that syntactic equivalence of structures preserves the polarity of a subcontext, although the structural depth of a context may vary under syntactic equivalence. An even number of negative inductive steps preserves the polarity of the base case, and an odd number of negative inductive steps inverts polarity. Positive steps preserve polarity. So an odd number of steps from the positive base case yields a negative context, and an even number of steps a positive context. An odd number of steps from the negative base case yields a positive context, and an even number of steps a negative context. We have two base cases, a positive and a negative one:

1. In the positive base case, $S\{\ \} = \{\ \}$, and is a positive context. It is immediate that $S\{R\} \models S\{T\}$. Note that the positions of R and T are preserved by this case.

2. In the negative base case, either $S\{\ \} = \langle\{\ \}; U\rangle$ or $S\{\ \} = \langle\{\ \}, U\rangle$, and is a negative context.

   (a) In the case that $S\{\ \} = \langle\{\ \}; U\rangle$, we have $S\{T\} = \langle T; U\rangle$. Suppose that $\mathfrak{M}, m \Vdash \langle T; U\rangle$. Then for any n such that $m \sqsubseteq n$, either $\mathfrak{M}, m \nVdash T$ or $\mathfrak{M}, m \Vdash U$, by the forcing clause for $\rightarrow$. At any such n, $\mathfrak{M}, n \nVdash R$, since $R \models T$, so $\mathfrak{M}, m \Vdash \langle R; U\rangle$, and hence $S\{T\} \models S\{R\}$.

   (b) In the case that $S\{\ \} = \langle\{\ \}, U\rangle$, we have $S\{T\} = \langle T, U\rangle$. Suppose that $\mathfrak{M}, m \Vdash \langle T, U\rangle$. Then for any n such that $\mathfrak{M}, n \Vdash T$, we have

$\mathfrak{M}, m \bullet n \Vdash U$, by the forcing clause for $\twoheadrightarrow$. Now suppose that $\mathfrak{M}, n' \Vdash R$ for some arbitrary n'. Then $\mathfrak{M}, n' \Vdash T$, because $R \vDash T$, and hence $\mathfrak{M}, m \bullet n' \Vdash U$. Then $\mathfrak{M}, m \Vdash \langle R, U\rangle$, and hence $S\{T\} \vDash S\{R\}$.

Note that the positions of R and T are switched by this case.

For the inductive step, we consider two groups of cases: those in which $S\{\ \}$ is a positive context; and those in which $S\{\ \}$ is a negative context. In each group we examine cases in which $S'\{\ \}$ (containing the same hole $\{\ \}$) is an immediate subcontext of $S\{\ \}$. In each inductive case we take $S'\{R\} \vDash S'\{T\}$ as the inductive hypothesis. We have seven cases for the inductive step.

1. First, we have five positive cases, each obtaining $S\{R\} \vDash S\{T\}$ from the hypothesis $S'\{R\} \vDash S'\{T\}$, where $S'\{\ \}$ occurs at a structural depth of one in $S\{\ \}$, and $S\{\ \}$ is a positive context. Note that the positions of R and T are preserved by each of these steps.

   (a) $S\{\ \} = (U; S'\{\ \}) \equiv (S'\{\ \}; U)$. Suppose that $(U; S'\{R\})$ is forced at state m in a model $\mathfrak{M}$, that is, $\mathfrak{M}, m \Vdash (U; S'\{R\})$. Then $\mathfrak{M}, m \Vdash U$ and $\mathfrak{M}, m \Vdash S'\{R\}$ by the forcing clause for $\wedge$. Since $S'\{R\} \vDash S'\{T\}$, $\mathfrak{M}, m \Vdash S'\{T\}$, and hence $\mathfrak{M}, m \Vdash (U; S'\{T\})$. So we have $(U; S'\{R\}) \vDash (U; S'\{T\})$, that is, $S\{R\} \vDash S\{T\}$.

   (b) $S\{\ \} = [U; S'\{\ \}] \equiv [S'\{\ \}; U]$. Suppose that $[U; S'\{R\}]$ is forced at state m in a model $\mathfrak{M}$, that is, $\mathfrak{M}, m \Vdash [U; S'\{R\}]$. Then either $\mathfrak{M}, m \Vdash U$ or $\mathfrak{M}, m \Vdash S'\{R\}$ by the forcing clause for $\vee$. In the case that $\mathfrak{M}, m \Vdash S'\{R\}$, we have $\mathfrak{M}, m \Vdash S'\{T\}$, since $S'\{R\} \vDash S'\{T\}$. So $\mathfrak{M}, m \Vdash S'\{T\}$, and hence $\mathfrak{M}, m \Vdash (U; S'\{T\})$. So we have $(U; S'\{R\}) \vDash (U; S'\{T\})$, that is, $S\{R\} \vDash S\{T\}$.

   (c) $S\{\ \} = \langle U; S'\{\ \}\rangle$. Suppose that $\langle U; S'\{R\}\rangle$ is forced at state m in a model $\mathfrak{M}$, that is, $\mathfrak{M}, m \Vdash \langle U; S'\{R\}\rangle$. Then for all n such that $m \sqsubseteq n$, either $\mathfrak{M}, n \nVdash U$ or $\mathfrak{M}, n \Vdash S'\{R\}$ by the forcing clause for $\rightarrow$. In cases where $\mathfrak{M}, n \Vdash S'\{R\}$, we have $\mathfrak{M}, n \Vdash S'\{T\}$, since $S'\{R\} \vDash S'\{T\}$. Hence $\mathfrak{M}, m \Vdash \langle U; S'\{T\}\rangle$, so we have $\langle U; S'\{R\}\rangle \vDash \langle U; S'\{T\}\rangle$, that is, $S\{R\} \vDash S\{T\}$.

(d) $S\{\ \} = (U, S'\{\ \}) \equiv (S'\{\ \}, U)$. Suppose that $(U, S'\{R\})$ is forced at state m in a model $\mathfrak{M}$, that is, $\mathfrak{M}, m \Vdash (U, S'\{R\})$. Then there exist n, n' such that $n \bullet n' \sqsubseteq m$, and $\mathfrak{M}, n \Vdash U$ and $\mathfrak{M}, n' \Vdash S'\{R\}'$ by the forcing clause for $*$. Then $\mathfrak{M}, n' \Vdash S'\{T\}$ since $S'\{R\} \vDash S'\{T\}$, and hence $\mathfrak{M}, m \Vdash (U, S'\{T\})$, so we have $(U, S'\{R\}) \vDash (U, S'\{T\})$, that is, $S\{R\} \vDash S\{T\}$.

(e) $S\{\ \} = \langle U, S'\{\ \}\rangle$. Suppose that $\langle U, S'\{R\}\rangle$ is forced at state m in a model $\mathfrak{M}$, that is, $\mathfrak{M}, m \Vdash \langle U, S'\{R\}\rangle$. Then for all n such that $\mathfrak{M}, n \Vdash U$, we have $\mathfrak{M}, m \bullet n \Vdash S'\{R\}$, by the forcing clause for $-\!*$. Since $S'\{R\} \vDash S'\{T\}$, we have $\mathfrak{M}, m \bullet n \Vdash S'\{T\}$ whenever $\mathfrak{M}, m \bullet n \Vdash S'\{R\}$, and hence $\mathfrak{M}, m \Vdash \langle U, S'\{T\}\rangle$, so we have $\langle U, S'\{R\}\rangle \vDash \langle U, S'\{T\}\rangle$, that is, $S\{R\} \vDash S\{T\}$.

Then we have two negative cases, each obtaining $S\{T\} \vDash S\{R\}$ from the hypothesis $S'\{R\} \vDash S'\{T\}$, where $S'\{\ \}$ occurs at a structural depth of one in $S\{\ \}$, and $S\{\ \}$ is a negative context. Note that the positions of R and T are switched by each of these steps.

(f) $S\{\ \} = \langle S'\{\ \}; U\rangle$. Suppose that $\langle S'\{T\}; U\rangle$ is forced at state m in a model $\mathfrak{M}$, that is, $\mathfrak{M}, m \Vdash \langle S'\{T\}; U\rangle$. Then for any n such that $m \sqsubseteq n$, either $\mathfrak{M}, m \nVdash S'\{T\}$ or $\mathfrak{M}, m \Vdash U$, by the forcing clause for $\rightarrow$. At any such n, $\mathfrak{M}, n \nVdash R$, since $S'\{R\} \vDash S'\{T\}$, so $\mathfrak{M}, m \Vdash \langle S'\{R\}; U\rangle$, and hence $S\{T\} \vDash S\{R\}$.

(g) $S\{\ \} = \langle S'\{\ \}, U\rangle$. Suppose that $\langle S'\{T\}, U\rangle$ is forced at state m in a model $\mathfrak{M}$, that is, $\mathfrak{M}, m \Vdash \langle S'\{T\}, U\rangle$. Then for any n such that $\mathfrak{M}, n \Vdash S'\{T\}$, we have $\mathfrak{M}, m \bullet n \Vdash U$, by the forcing clause for $-\!*$. Now suppose that $\mathfrak{M}, n' \Vdash S'\{R\}$ for some arbitrary n'. Then $\mathfrak{M}, n' \Vdash S'\{T\}$, by the inductive hypothesis, and hence $\mathfrak{M}, m \bullet n' \Vdash U$. So $\mathfrak{M}, m \Vdash \langle S'\{R\}, U\rangle$, and hence $S\{T\} \vDash S\{R\}$. ∎

This proof attests to the non-interference of the two implicative connectives when they are nested – at least from the point-of-view of soundness. The result is evidence for the correctness of our two-valued scheme for the polarities of contexts.

Theorem 2 (Soundness) *If $R \vdash T$ then $R \vDash T$. That is, the rules of inference and rules of syntactic equivalence of* SBISg *preserve forcing at any state in any model in elementary Kripke resource semantics.* □

PROOF A rule application occurs either in a positive or a negative context. Positive rules may be applied in positive contexts and negative rules in negative contexts. To show soundness, we show that each positive rule of inference preserves validity in a model when applied in a top-level context, and each negative rule when applied in a shallow negative context, that is, a context with a structural depth of one.

1. *Positive rules.*

   (a) im$\downarrow$ Suppose that $\mathfrak{M}, m \Vdash I$. Then by the forcing clause for I, $e \sqsubseteq m$. Suppose further that for some arbitrary n and R, $\mathfrak{M}, n \Vdash R$. Then since $n = e \bullet n$, we have R, $\mathfrak{M}, e \bullet n \Vdash R$. Hence $\mathfrak{M}, e \Vdash \langle R, R\rangle$ for arbitrary R, by the forcing clause for $-\!\!*$. Then by generalised monotonicity (Lemma 1) we have $\mathfrak{M}, m \Vdash \langle R, R\rangle$ for all m such that $e \sqsubseteq m$. Hence $I \vDash \langle R, R\rangle$.

   (b) scm$\downarrow$ Suppose that $\mathfrak{M}, m \Vdash (\langle R, T\rangle, U)$. Then by the forcing clause for $*$, there exist n, n' such that $n \bullet n' \sqsubseteq m$, and $\mathfrak{M}, n \Vdash \langle R, T\rangle$ and $\mathfrak{M}, n' \Vdash U$, and then by the forcing clause for $-\!\!*$, for all m' such that $\mathfrak{M}, m' \Vdash R$, we have $\mathfrak{M}, n \bullet m' \Vdash T$. Then for all m' such that $\mathfrak{M}, m' \Vdash R$, by bifunctoriality there exist $n \bullet m', n'$ such that $n \bullet n' \bullet m' \sqsubseteq m \bullet m'$ and $\mathfrak{M}, n \bullet m' \Vdash T$ and $\mathfrak{M}, n' \Vdash U$, that is $\mathfrak{M}, m \bullet m' \Vdash (T, U)$. Hence $\mathfrak{M}, m \Vdash \langle R, (T, U)\rangle$, and so $(\langle R, T\rangle, U) \vDash \langle R, (T, U)\rangle$.

   (c) sim$\downarrow$ Suppose that $\mathfrak{M}, m \Vdash (R, \langle T, U\rangle)$. Then by the forcing clause for $*$, there exist n, n' such that $n \bullet n' \sqsubseteq m$ and $\mathfrak{M}, n \Vdash R$ and $\mathfrak{M}, n' \Vdash \langle T, U\rangle$, and then by the forcing clause for $-\!\!*$, for all m' such that $\mathfrak{M}, m' \Vdash T$, $\mathfrak{M}, n' \bullet m' \Vdash U$. Now suppose that $\mathfrak{M}, m'' \Vdash \langle R, T\rangle$ for arbitrary m''. Then for all n'' such that $\mathfrak{M}, n'' \Vdash R$ we have $\mathfrak{M}, m'' \bullet n'' \Vdash T$, by the forcing clause for $-\!\!*$. Then we have $\mathfrak{M}, n' \bullet m'' \bullet n'' \Vdash U$, and hence for all n'', $\mathfrak{M}, n' \bullet n'' \Vdash \langle\langle R, T\rangle, U\rangle$, by the forcing clause for $-\!\!*$, and thus $\mathfrak{M}, n \bullet n' \Vdash \langle\langle R, T\rangle, U\rangle$, by instantiating n'' with n.

By generalised monotonicity (Lemma 1), $\mathfrak{M}, m \Vdash \langle\langle R, T\rangle, U\rangle$, and so $(R, \langle T, U\rangle) \vDash \langle\langle R, T\rangle, U\rangle$.

(d) ia$\downarrow$ Suppose that $\mathfrak{M}, m \Vdash \top$. Then we have to consider all m, by the forcing clause for $\top$. For all n, and so all n such that for any m, $m \sqsubseteq n$, and arbitrary R, either $\mathfrak{M}, n \nVdash R$ or $\mathfrak{M}, n \Vdash R$, and so by the forcing clause for $\rightarrow$, $\top \vDash \langle R; R\rangle$.

(e) sca$\downarrow$ Suppose that $\mathfrak{M}, m \Vdash (\langle R; T\rangle; \langle U; V\rangle)$. Then by the forcing clauses for $\rightarrow$ and $\wedge$, we have for all n such that $m \sqsubseteq n$, either $\mathfrak{M}, n \nVdash R$ or $\mathfrak{M}, n \Vdash T$, and either $\mathfrak{M}, n \nVdash U$ or $\mathfrak{M}, n \Vdash V$. So either $\mathfrak{M}, n \nVdash R$ and $\mathfrak{M}, n \nVdash U$, or $\mathfrak{M}, n \nVdash R$ and $\mathfrak{M}, n \Vdash V$, or $\mathfrak{M}, n \Vdash T$ and $\mathfrak{M}, n \nVdash U$, or $\mathfrak{M}, n \Vdash T$ and $\mathfrak{M}, n \Vdash V$. So either not $\{\mathfrak{M}, n \Vdash R$ and $\mathfrak{M}, n \Vdash U\}$ (first three cases) or $\mathfrak{M}, n \Vdash T$ and $\mathfrak{M}, n \Vdash V$ (last case). Then by the forcing clauses for $\rightarrow$ and $\wedge$, we have $\mathfrak{M}, m \Vdash \langle (R; U); (T; V)\rangle$ and hence $(\langle R; T\rangle; \langle U; V\rangle) \vDash \langle (R; U); (T; V)\rangle$.

(f) sda$\downarrow$ Suppose that $\mathfrak{M}, m \Vdash (\langle R; T\rangle; \langle U; V\rangle)$. Then by the forcing clauses for $\rightarrow$ and $\wedge$, we have for all n such that $m \sqsubseteq n$, either $\mathfrak{M}, n \nVdash R$ or $\mathfrak{M}, n \Vdash T$, and either $\mathfrak{M}, n \nVdash U$ or $\mathfrak{M}, n \Vdash V$. So either $\mathfrak{M}, n \nVdash R$ and $\mathfrak{M}, n \nVdash U$, or $\mathfrak{M}, n \nVdash R$ and $\mathfrak{M}, n \Vdash V$, or $\mathfrak{M}, n \Vdash T$ and $\mathfrak{M}, n \nVdash U$, or $\mathfrak{M}, n \Vdash T$ and $\mathfrak{M}, n \Vdash V$. So either not $\{\mathfrak{M}, n \Vdash R$ or $\mathfrak{M}, n \Vdash U\}$ (the first case) or $\{\mathfrak{M}, n \Vdash T$ or $\mathfrak{M}, n \Vdash V\}$ (remaining cases). Then by the forcing clauses for $\rightarrow$ and $\vee$, we have $\mathfrak{M}, m \Vdash \langle [R; U]; [T; V]\rangle$ and hence $(\langle R; T\rangle; \langle U; V\rangle) \vDash \langle [R; U]; [T; V]\rangle$.

(g) sia$\downarrow$ Suppose that $\mathfrak{M}, m \Vdash (\langle R; T\rangle; \langle U; V\rangle)$. Then by the forcing clauses for $\rightarrow$ and $\wedge$, we have for all n such that $m \sqsubseteq n$, either $\mathfrak{M}, n \nVdash R$ or $\mathfrak{M}, n \Vdash T$, and either $\mathfrak{M}, n \nVdash U$ or $\mathfrak{M}, n \Vdash V$. So either $\mathfrak{M}, n \nVdash R$ and $\mathfrak{M}, n \nVdash U$, or $\mathfrak{M}, n \nVdash R$ and $\mathfrak{M}, n \Vdash V$, or $\mathfrak{M}, n \Vdash T$ and $\mathfrak{M}, n \nVdash U$, or $\mathfrak{M}, n \Vdash T$ and $\mathfrak{M}, n \Vdash V$. So either not $\{$either $\mathfrak{M}, n \nVdash T$ or $\mathfrak{M}, n \Vdash U\}$ (i.e. $\mathfrak{M}, n \Vdash T$ and $\mathfrak{M}, n \nVdash U$, the third case) or $\{$either $\mathfrak{M}, n \nVdash R$ or $\mathfrak{M}, n \Vdash V\}$ (remaining cases). Then by the forcing clause for $\rightarrow$ (twice), we have $\mathfrak{M}, m \Vdash \langle\langle R; U\rangle; \langle T; V\rangle\rangle$ and hence $(\langle R; T\rangle; \langle U; V\rangle) \vDash \langle\langle R; U\rangle; \langle T; V\rangle\rangle$.

(h) cl$\uparrow$ Suppose that $\mathfrak{M}, m \Vdash R$. Then by the forcing clause for $\wedge$,

$\mathfrak{M}, m \Vdash R$ and $\mathfrak{M}, m \Vdash R$, so we have $R \vDash (R; R)$

(i) wl $\uparrow$ Suppose that $\mathfrak{M}, m \Vdash R$. By the forcing clause for $\top$, $\mathfrak{M}, m \Vdash \top$, so we have $R \vDash \top$.

(j) cr$\downarrow$ Suppose that $\mathfrak{M}, m \Vdash [R; R]$. Then by the forcing clause for $\vee$, either $\mathfrak{M}, m \Vdash R$ or $\mathfrak{M}, m \Vdash R$, that is $\mathfrak{M}, m \Vdash R$, so $[R; R] \vDash R$.

(k) wr$\downarrow$ By the forcing clause for $\bot$, $\mathfrak{M}, m \nVdash \bot$ for any m. Then for any m, either $\mathfrak{M}, m \nVdash \bot$ or $\mathfrak{M}, m \Vdash R$ for an arbitrary R, so by Definition 25, we have $\bot \vDash R$.[14]

2. *Negative rules.* For each negative rule $\rho \dfrac{R}{T}$ we have to show that it preserves forcing at an arbitrary state in and arbitrary model, just in the two shallowest negative contexts $\langle\{\ \}; U\rangle$ and $\langle\{\ \}, U\rangle$. For each rule, we have already established the semantic entailment $T \vdash R$ in the case of its positive dual rule, so we can treat the two shallow contexts generically.

   (a) Suppose that $\mathfrak{M}, m \Vdash \langle R; U\rangle$. Then for all n such that $m \sqsubseteq n$, either $\mathfrak{M}, n \nVdash R$ or $\mathfrak{M}, n \Vdash U$. Either $\mathfrak{M}, n \nVdash T$ or $\mathfrak{M}, n \Vdash U$. Hence $\mathfrak{M}, m \Vdash \langle T; U\rangle$, and so $\langle R; U\rangle \vDash \langle T; U\rangle$.

   (b) Suppose that $\mathfrak{M}, m \Vdash \langle R, U\rangle$. Then for all n such that $\mathfrak{M}, n \Vdash R$, we have $\mathfrak{M}, m \bullet n \Vdash U$. Since we have already shown $T \vDash R$ in the positive case for each rule, for all n such that $\mathfrak{M}, n \Vdash T$, we have $\mathfrak{M}, m \bullet n \Vdash U$. Hence $\mathfrak{M}, m \Vdash \langle T, U\rangle$, and so $\langle R, U\rangle \vDash \langle T, U\rangle$.

Then, by the semantic depth lemma (Lemma 9), we have that each rule of inference preserves forcing at any state in any model, when applied in a context of appropriate polarity, at arbitrary depth.

We omit the treatment of the rules of syntactic equivalence, which is straightforward and similar to that above. Note that each rule of syntactic equivalence needs to be treated bidirectionally, and in both positive and negative contexts. ∎

[14] For the new Kripke resource semantics, this case runs as follows. Suppose that $\mathfrak{M}, m \Vdash \bot$. Then by the forcing clause for $\bot$, $m = \pi$. Since $\mathfrak{M}, \pi \Vdash R$ for arbitrary R, by Lemma 2, we have $\bot \vDash R$.

Corollary 1 *If* $U \vdash \bot$ *then for any* m*,* $\mathfrak{M}, m \not\Vdash U$*.* □

PROOF Suppose that $U \vdash \bot$. Then by soundness (Theorem 2) $U \Vdash \bot$, that is, at every state m in $\mathfrak{M}$ at which U is forced, $\bot$ is forced. But by the forcing clause for $\bot$, $\bot$ is never forced, and hence $\mathfrak{M}, m \not\Vdash U$ for all m. ■

3.4.3. Digression: classical consistency

We present two standard and equivalent classical definitions of consistency, which we call $\bot$-consistency and $\exists$-consistency. We use the setting of SBISg. This is principally for the purpose of illustration. Classical definitions of consistency are not adequate for the kind of model existence result we would require for BI, or for intuitionistic logic, if we followed directly the standard strategy for modal logic. Lemmas 10, 11, 12 and 13 do hold for SBISg, however.

Definition 29 A set of structures Γ is $\bot$-*inconsistent* if $\bot$ is deducible from Γ, otherwise it is $\bot$-*consistent*. A structure R is $\bot$-inconsistent if $\{R\}$ is $\bot$-inconsistent, otherwise R is $\bot$-consistent. □

Lemma 10 *If* Γ *is* $\bot$*-inconsistent, then any* Δ*, such that* $\Gamma \subseteq \Delta$*, is* $\bot$*-inconsistent.* □

PROOF Since Γ is $\bot$-inconsistent, there is some Γ' such that $\Gamma' \subseteq \Gamma$, and $\vdash \langle \Gamma'; \bot \rangle$. Since $\Gamma' \subseteq \Delta$, Δ is $\bot$-inconsistent. ■

Lemma 11 *If* Γ *is* $\bot$*-consistent, then any* Γ' *such that* $\Gamma' \subseteq \Gamma$ *is* $\bot$*-consistent.* □

PROOF We argue by contraposition. Suppose that Γ' were $\bot$-inconsistent. Then $\vdash \langle \Gamma''; \bot \rangle$ for some Γ'' such that $\Gamma'' \subseteq \Gamma'$. But $\Gamma'' \subseteq \Gamma$, so Γ would be $\bot$-inconsistent. ■

Lemma 12 *If* Γ *is* $\bot$*-consistent, and* R *is deducible from some* Γ' *such that* $\Gamma' \subseteq \Gamma$*, then* $\Gamma \cup \{R\}$ *is* $\bot$*-consistent.* □

PROOF We argue by contraposition. Suppose that $\Gamma \cup \{R\}$ is $\bot$-inconsistent, then either Γ is $\bot$-inconsistent, or there is some Γ'' such that $\Gamma'' \subseteq \Gamma$ and $\vdash \langle \Gamma''; \langle R; \bot \rangle \rangle$. Since $\Gamma' \vdash R$, we have a derivation of $\langle \Gamma; \bot \rangle$ from $\langle \Gamma''; \langle R; \bot \rangle \rangle$:

$$
\begin{array}{rcl}
 & \vdots & \\ \cline{2-2}
 & \langle \Gamma''; \langle R; \bot \rangle \rangle & \\ \cline{2-2}
\mathsf{wl}\downarrow & \langle \Gamma; \langle R; \bot \rangle \rangle & \\ \cline{2-2}
\equiv & (\langle \Gamma; \langle R; \bot \rangle \rangle; \top) & \\ \cline{2-2}
 & \Downarrow & \\ \cline{2-2}
 & (\langle \Gamma; \langle R; \bot \rangle \rangle; \langle \Gamma'; R \rangle) & \text{Paste } \begin{array}{c} \top \\ \| \\ \langle \Gamma'; R \rangle \end{array} \\ \cline{2-2}
\mathsf{wl}\downarrow & (\langle \Gamma; \langle R; \bot \rangle \rangle; \langle \Gamma; R \rangle) & \\ \cline{2-2}
\mathsf{sca}\downarrow & \langle (\Gamma; \Gamma); (\langle R; \bot \rangle; R) \rangle & \\ \cline{2-2}
\mathsf{cl}\downarrow & \langle \Gamma; (\langle R; \bot \rangle; R) \rangle & \\ \cline{2-2}
\equiv & \langle \Gamma; (\langle \top; R \rangle; \langle R; \bot \rangle) \rangle & \\ \cline{2-2}
\mathsf{sia}\downarrow & \langle \Gamma; \langle \langle R; R \rangle; \langle \top; \bot \rangle \rangle \rangle & \\ \cline{2-2}
\mathsf{ia}\uparrow & \langle \Gamma; \langle \top; \langle \top; \bot \rangle \rangle \rangle & \text{(cut)} \\ \cline{2-2}
\equiv & \langle \Gamma; \bot \rangle &
\end{array}
$$

So $\vdash \langle \Gamma; \bot \rangle$, and hence Γ is $\bot$-inconsistent. ∎

Remark 6 The following classical lemma does not hold for SBISg:

$$\text{If } \nvdash R \text{ then } \langle R; \bot \rangle \text{ is } \bot\text{-consistent}$$

Essentially, this is due to the inadmissibility of a rule of double-negation elimination. □

Definition 30 A set of structures Γ is $\exists$-*consistent* if there exists some structure R which is not deducible from Γ, otherwise it is $\exists$-*inconsistent*. □

Lemma 13 *A set of structures Γ is $\exists$-consistent iff it is $\bot$-consistent.* □

Proof Left to right, then right to left:

1. If Γ is $\exists$-consistent, then there is some R such that $\nvdash \langle \Gamma; R \rangle$. Suppose that Γ is $\bot$-inconsistent. Then $\vdash \langle \Gamma; \bot \rangle$, and consequently $\vdash \langle \Gamma; R \rangle$ for arbitrary R, using rule $\mathsf{wr}\downarrow$, which is a contradiction. Hence Γ is $\bot$-consistent.

2. If Γ is $\bot$-consistent, then $\nvdash \langle \Gamma, \bot \rangle$. Suppose that Γ is $\exists$-inconsistent, then $\vdash \langle \Gamma, R \rangle$ for any R, and hence $\vdash \langle \Gamma, \bot \rangle$, which is a contradiction. Hence Γ is $\exists$-consistent. ∎

3.4.4. Completeness

We now prove completeness with respect to the PDM semantics for BI. Like LBI with $\perp$, SBISg with $\perp$ is incomplete with respect to the elementary Kripke semantics. This incompleteness is the motivation behind the development of the various semantic variations: initially with the topological semantics, and later on relational and new Kripke semantics with π, and the PDM semantics. Essentially, the difficulty revolves around the fact that it is proof-theoretically possible for the multiplicative conjunction of a pair of consistent formulæ (or bunches, or structures) to be inconsistent. That is, it may be the case that $R \nvdash \perp$ and $T \nvdash \perp$, but that $\langle R, T \rangle \vdash \perp$. The usual example is $p * (p \mathrel{-\!\!*} \perp) \vdash \perp$, or $(a, \langle a, \perp \rangle) \vdash \perp$ in the language of structures, where we have $a \nvdash \perp$ and $\langle a, \perp \rangle \nvdash \perp$.[15]

Proposition 3 *In the elementary Kripke resource semantics, for any state m of any model $\mathfrak{M}$, and structure R, $\mathfrak{M}, m \Vdash \langle\langle R, \perp \rangle; \perp \rangle$ iff there exists $n \in M$ such that $\mathfrak{M}, n \Vdash R$. That is, $\langle\langle R, \perp \rangle; \perp \rangle$ is satisfied in a model iff R is satisfied in that model.* (This is noted as routine during the proof of Pym's [2002] Proposition 4.8.) $\square$

PROOF Consider the elementary forcing clauses for $\mathrel{-\!\!*}$ and $\perp$. First, we observe that $\mathfrak{M}, m \Vdash \langle R, \perp \rangle$ iff there is no $m \bullet n$, and hence no n such that $\mathfrak{M}, n \Vdash R$.[16] Now suppose that $\mathfrak{M}, m \Vdash \langle\langle R, \perp \rangle; \perp \rangle$. Then by the forcing clauses for $\to$ and $\perp$, there is no $n \in M$ such that $m \sqsubseteq n$ and $\mathfrak{M}, n \Vdash \langle R, \perp \rangle$. Then $\mathfrak{M}, m \nVdash \langle R, \perp \rangle$, and hence there is some n such that $\mathfrak{M}, n \Vdash R$. Now take the right to left case. Suppose that there is some n such that $\mathfrak{M}, n \Vdash R$. Then for any m', $\mathfrak{M}, m' \nVdash \langle R, \perp \rangle$, and hence for any m, $\mathfrak{M}, m \Vdash \langle\langle R, \perp \rangle; \perp \rangle$ by the forcing clauses for $\to$ and $\perp$. $\blacksquare$

Pym [2002, Proposition 4.8][17] gives an example of a semantic entailment in the elementary Kripke resource semantics for which there is no corresponding proof in LBI. This can be readily seen for cut-free LBI. Rewritten in the calculus of

[15] On this point, see Pym, O'Hearn & Yang [2004, §§3.5, 5.2].

[16] In the elementary semantics. In the PDM semantics, it means $m \bullet n$ is never defined when $\mathfrak{M}, n \Vdash R$; and in the new Kripke resource semantics, it means that $m \bullet n = \pi$ for any n such that $\mathfrak{M}, n \Vdash R$. Recall that in the new Kripke resource semantics, $m \sqsubseteq \pi$ for any m and $m = \pi$ iff $\mathfrak{M}, m \Vdash \perp$, rather than $\mathfrak{M}, m \nVdash \perp$ for all m as in the elementary and PDM semantics.

[17] Also Pym, O'Hearn & Yang [2004, §3.5, Proposition 6]

structures, for any structures R and T:

$$(\langle\langle R, \bot\rangle; \bot\rangle; \langle\langle T, \bot\rangle; \bot\rangle) \vDash \langle\langle (R, T), \bot\rangle; \bot\rangle$$

This works as follows. If $\mathfrak{M}, m \Vdash \langle\langle R, \bot\rangle; \bot\rangle$ then for some n, $\mathfrak{M}, n \Vdash R$. Similarly we obtain $\mathfrak{M}, n' \Vdash T$ for some n'. Then in the elementary semantics, $n \bullet n'$ is defined,[18] and hence by the forcing clause for $*$, we have $\mathfrak{M}, n \bullet n' \Vdash (R, T)$.[19] Then by Proposition 3, $\mathfrak{M}, m \Vdash \langle\langle (R, T), \bot\rangle; \bot\rangle$. Indeed, this follows for any m, and hence for the m we started with. But, substituting atoms a, b for arbitrary structures R, T:

$$(\langle\langle a, \bot\rangle; \bot\rangle; \langle\langle b, \bot\rangle; \bot\rangle) \nvDash \langle\langle (a, b), \bot\rangle; \bot\rangle.$$

Loosely speaking, we can readily see that the switch rules of SBISg offer no way for atoms a and b to cross an additive boundary to become multiplicatively conjoined in the course of a derivation. And we can readily see – at least for cut-free BISg – that there is no way for a or b or (a, b) to 'materialise' in a multiplicative context, since no structural rules are accessible in a multiplicative context. By "not accessible", we mean that for a structure R in an arbitrary context, the unit rules of syntactic equivalence can introduce $\top$ or $\bot$ only in additive structures, and eliminate them only from additive structures.

The strategy of our completeness proof is similar to that used for the standard completeness proofs for propositional NBI without $\bot$ with respect to the elementary Kripke semantics and, for NBI with respect to the topological Kripke semantics [Pym 2002, §4.2, §5.2] (also see Pym, O'Hearn & Yang [2004]), and for the semantic tableau system TBI with respect to the relational semantics [Galmiche, Méry & Pym 2005, §5]. These are all essentially similar in strategy to the completeness proof for intuitionistic logic given by van Dalen [2004, §5.3], in the reliance upon a countermodel construction and the use of prime theories. We an indebted to all of these presentations. Our formulation and use of prime theories

[18]This step could not be made with the PDM semantics.

[19]Just for the moment, we skirt around the requirement that $n \bullet n' \in M$, and assume that it is. It is conceivable that $n \bullet n' \notin M$ and that there is no m such that $n \bullet n' \sqsubseteq m$, but in the apparatus of prime theories that follows, it will always be that case that if $n \bullet n' \downarrow$ then there is some $m \in M$ such that $n \bullet n' \sqsubseteq m$.

is considerably simpler than the use of prime bunches found in other treatments of BI. Proof with respect to the PDM semantics also permits certain simplification, for instance, in the definition of the • operation in a canonical countermodel. Moreover, the calculus of structures allows certain simplifications, since we do not have to worry about the distinctions between formulæ, bunches and sequents.

Our early attempts at the proof were in fact not inspired by these proofs, but by the standard strategy for completeness proofs for modal logics with respect to standard Kripke semantics for modal logics [Blackburn et al. 2001, chapter 4]. We are in fact indebted to Blackburn et al.'s [2001] presentation. The principal difference between proofs for modal logic and for BI or intuitionistic logic is that since standard modal logics are essentially classical, a notion of consistency is available. No appropriate notion of consistency is available for BI or for intuitionistic logic. Refer to our presentation in Section 3.4.3 of a classical notion of consistency for SBISg that is not adequate for our purpose here.

For modal logic, it is straightforward to give a proof of the theorem that a proof system is complete with respect to a semantics iff every consistent set of formulæ is satisfiable in the semantics. This is essentially due to the applicability of a standard classical definition of a consistent set of formulæ and the availability (or admissibility) of a rule of double negation-elimination in proof systems for modal logic – luxuries we have to do without. It is shown that any consistent set of formulæ may be extended to a maximally consistent set of formulæ. A canonical Kripke model is then constructed in which the states are maximally consistent sets of formulæ. It is shown that a formula is forced at a state in the model iff it is a member of the maximally consistent set at that state. The proof of completeness follows from this.

We did examine the possibility of adapting the notion of a *consistency property* for intuitionistic logic, which is used in a completeness proof to compensate for the lack of a natural notion of consistency in intuitionistic logic [Fitting 1973, Fitting 1983, chapter 9, §7]. This notion is well-suited for a completeness proof for a tableau system. The idea of a consistency property uses signed formulæ, which are a familiar feature of tableau systems for BI and intuitionistic logic. It also depends crucially on the decomposition of signed formulæ into pairs of signed formulæ, paired conjunctively or disjunctively. There seemed to be some natural connection between the signs of formulæ and the polarity of contexts in SBISg – negative sign or polarity corresponds to the antecedent position in a conditional. Polarity and signed formulæ are both devices that compensate for the lack of involutive negation. But nothing concretely useful arose in this connection. It is the 'mingling' of signed subformulæ, and eventually signed atoms, of formulæ which does the work of a consistency property, but in SBISg, rules of inference preserve polarity, and there is no systematic decomposition of structures to atomic structures.

In the standard completeness proof for NBI, and the proof for intuitionistic logic, the rôle of consistent sets – a set of formulæ is consistent if $\perp$ is not deducible from it – is taken by sets of formulæ Γ from which a given formula ϕ is not deducible. Although this does not permit the construction of a canonical model for every provable formula, it does permit the construction of especially tailored canonical countermodels. It can then be shown that if $\Gamma \nvdash \phi$ then $\Gamma \nvDash \phi$, which amounts to completeness. It is sufficient, on the supposition that $R \nvdash T$, to construct a single model $\mathfrak{M}$ (in any frame) such that for some state $m \in M$, we have $\mathfrak{M}, m \Vdash R$ and $\mathfrak{M}, m \nVdash T$, and hence that $R \nvDash T$. These proofs use the notion of a prime bunch or prime theory, in place of a maximally consistent set. We have opted for the construction of maximal prime theories in the style of Lindenbaum's Lemma, similar to Blackburn et al. [2001, Lemma 4.17] and van Dalen [2004, Lemma 5.3.8], rather than the complicated system of prime evaluation used in the proof for NBI. We now proceed towards the completeness result.

Remark 7 *Equivocation over syntactic equivalence.* Our general rule is to use syntactic equivalence implicitly in proof-theoretical considerations, but always to treat it explicitly in semantic considerations. A syntactic equivalence should be regarded as a rule of inference that is invertible and applies in contexts of either polarity. □

Remark 8 At some points we safely equivocate between sets of formulæ or structures $\{R_1, \ldots, R_n\}$, and additive conjunctive structures, that is, structures of the form $(R_1; \ldots; R_n)$. □

We now introduce the idea of a prime theory, which plays a pivotal rôle in our completeness proof. Our definition is essentially that of van Dalen [2004, Definition 5.3.7], in his proof of the completeness of intuitionistic logic. We then proceed to give a prime theory existence lemma, based upon what is essentially a Henkin construction. Our procedure is close to that of van Dalen [2004, Lemma 5.3.8].

Definition 31 A *prime theory* is a set Σ of structures that meets the following conditions:

(i) Σ is consistent;

(ii) Σ is closed under deducibility. That is, for any R, if $\Sigma \vdash R$ then $R \in \Sigma$;

(iii) If $[R; T] \in \Sigma$ then $R \in \Sigma$ or $T \in \Sigma$.

Note that trivially, if $R \in \Sigma$ then $\Sigma \vdash R$, and hence $\Sigma \vdash R$ iff $R \in \Sigma$. □

Lemma 14 *A prime theory contains every theorem.* □

PROOF If T is a theorem, then $\Sigma \vdash T$ for any set of structures Σ. Hence by Definition 31, if Σ^+ is a prime theory, then $T \in \Sigma^+$. ■

Proposition 4 *The structures of* BI *are enumerable.* □

Lemma 15 (Prime theory existence lemma) *For any structures R, T such that $R \nvdash T$ there is a prime theory Σ^+ such that $R \in \Sigma^+$ and $\Sigma^+ \nvdash T$ (and hence $T \notin \Sigma^+$ since Σ^+ is a prime theory).* □

PROOF Given structures R, T such that $R \nvdash T$, we may construct a prime theory Σ^+ such that $\Sigma^+ \nvdash T$. Let $R_1, R_2, R_3, \ldots$ be an enumeration of the structures of BI.[20] Construct Σ^+ as follows:

[20] We could indeed filter the enumeration so that $\Sigma_n \vdash R_n$ at each step, as does van Dalen [2004, Lemma 5.3.8]. This would simplify slightly the treatment of disjunctive structures, and exclude unneeded 'irrelevant' structures from the prime theory. The filtered enumeration would in any case still be infinite in size. On the whole, however, this is unnecessary and would complicate our treatment.

$$\Sigma_0 \quad = \quad \{R\}$$

If R_i is of disjunctive form $[R_L; R_R]$

$$\Sigma_{i+1} \quad = \quad \begin{cases} \Sigma_i \cup \{R_i, R_L\} & \text{if } \Sigma_i \cup \{R_L\} \nvdash T\text{; otherwise} \\ \Sigma_i \cup \{R_i, R_R\} & \text{if } \Sigma_i \cup \{R_R\} \nvdash T \\ \Sigma_i & \text{otherwise} \end{cases}$$

otherwise

$$\Sigma_{i+1} \quad = \quad \begin{cases} \Sigma_i \cup \{R_i\} & \text{if } \Sigma_i \cup \{R_i\} \nvdash T \\ \Sigma_i & \text{otherwise} \end{cases}$$

and finally

$$\Sigma^+ \quad = \quad \bigcup_{i \geq 0} \Sigma_i$$

Σ^+ is the chain union of successive sets Σ_i. Note that $\Sigma_i \subseteq \Sigma_{i+1}$ for any finite i, and consequently $\bigcup_{i=0}^{j} \Sigma_i = \Sigma_j$ for any finite j. So $\Sigma_i \subseteq \Sigma^+$ for any i. Crudely speaking, we can say that $\Sigma^+ = \Sigma_\omega$. It remains to show that $\Sigma^+ \nvdash T$ and that Σ^+ is indeed a prime theory.

1. T is not deducible from Σ_0 since $\Sigma_0 = \{R\}$ and $\{R\} \nvdash T$. The Σ_{i+1} step preserves non-deducibility of T, so by induction, $\Sigma_i \nvdash T$ for any finite i. Hence $\Sigma^+ \nvdash T$.

2. We check that Σ^+ meets the necessary conditions to be a prime theory.

    (a) Suppose that there is some U such that $U \notin \Sigma^+$ and $\Sigma^+ \vdash U$. Trivially, if $U = R$ then $U \in \Sigma^+$, so if $U \notin \Sigma^+$ then it is because there is some i such that $\Sigma_i \cup \{U\} \vdash T$, or in the case that $U = [U_L; U_R]$, some i such that $\Sigma_i \cup \{U_L\} \vdash T$ and $\Sigma_i \cup \{U_R\} \vdash T$. Now given that $\Sigma^+ \vdash U$, there is some Σ' of finite size such that $\Sigma' \subseteq \Sigma^+$ and $\vdash \langle \Sigma'; U \rangle$. Hence, we can construct a proof of $\langle \Sigma_i; \Sigma'; T \rangle$

$$
\begin{array}{c}
\top \\ \hline
\vdots \\ \hline
\langle(\Sigma_i; U); T\rangle \\ \hline
\Uparrow \\ \hline
\langle(\Sigma_i; \Sigma'); T\rangle
\end{array}
\ \text{Paste negative of}\ \begin{array}{c}\Sigma' \\ \| \\ U\end{array}
$$

or in the case of disjunctive U

$$
\begin{array}{rcl}
 & \top & \\ \cline{2-2}
 & \vdots & \\ \cline{2-2}
 & (\langle(\Sigma_i; U_L); T\rangle; \langle(\Sigma_i; U_R); T\rangle) & \\ \cline{2-2}
\equiv & & \text{(currying, twice)} \\
 & (\langle\Sigma_i; \langle U_L; T\rangle\rangle; \langle\Sigma_i; \langle U_R; T\rangle\rangle) & \\ \cline{2-2}
\mathsf{sca}\!\downarrow & & \\
 & \langle(\Sigma_i; \Sigma_i); (\langle U_L; T\rangle; \langle U_R; T\rangle)\rangle & \\ \cline{2-2}
\mathsf{cl}\!\downarrow, \mathsf{sda}\!\downarrow & & \\
 & \langle\Sigma_i; \langle[U_L; U_R]; [T; T]\rangle\rangle & \\ \cline{2-2}
\equiv, \mathsf{cr}\!\downarrow & & \text{(uncurrying)} \\
 & \langle(\Sigma_i; [U_L; U_R]); T\rangle & \\ \cline{2-2}
 & \Uparrow & \\ \cline{2-2}
 & \langle(\Sigma_i; \Sigma'); T\rangle & \text{Paste negative of}\ \begin{array}{c}\Sigma' \\ \| \\ U\end{array}
\end{array}
$$

Since $\Sigma_n \subseteq \Sigma^+$ and $\Sigma' \subseteq \Sigma^+$, we have $\Sigma_n \cup \Sigma' \subseteq \Sigma^+$, and hence $\Sigma^+ \vdash T$, yielding a contradiction. Hence $\Sigma^+ \nvdash U$ for any U such that $U \notin \Sigma^+$. By contraposition, if $\Sigma^+ \vdash U$ then $U \in \Sigma^+$, that is, Σ^+ is closed under $\vdash$.

(b) By the construction of the Σ_{n+1} step, if $[U_1; U_2] \in \Sigma^+$, then either $U_1 \in \Sigma^+$ or $U_2 \in \Sigma^+$. ■

Remark 9 Membership of Σ^+ depends of the ordering of the enumeration. For instance, suppose that the atomic structure a occurs before the structure $\langle a; T\rangle$ in the enumeration. Then $a \in \Sigma^+$ and $\langle a; T\rangle \notin \Sigma^+$. But if the order is reversed, $\langle a; T\rangle \in \Sigma^+$ and $a \notin \Sigma^+$. □

Remark 10 Any structure T may be rewritten $\langle \top; T\rangle$ or $\langle I, T\rangle$ according to the syntactic equality of structures specified in Figure 3.1. Hence we can construct a prime theory from $\nvdash T$ using $\Sigma_0 = \{\top\}$. □

Lemma 16 (Prime extension lemma) *For any set of structures Σ and structure T such that $\Sigma \nvdash T$, there is a prime theory Σ^+ such that $\Sigma \subseteq \Sigma^+$ and $\Sigma^+ \nvdash T$.* (Cf. the extension lemma of Galmiche et al. [2005, Lemma 5.2]). □

PROOF The proof is a straightforward variation of the proof for Lemma 15. ■

We now show that the intersection of all prime extensions of a set Σ represents the deductive strength or content of Σ in a precise way.[21] It contains all and only the structures that are deducible from Σ, and it is *almost* a prime theory. It does not contain extraneous or irrelevent elements, as we see in the prime theories constructed according to the method used in the proof of Lemma 15, in which arbitary structures are drawn from an enumeration of the structures of BI and are tested systematically for inclusion in a prime theory, constructed as a limit construction.

Lemma 17 (Prime extension intersection lemma) *Let Σ be a consistent set of structures, and let $\bigcap \Sigma^+$ be the intersection of all prime extensions of Σ. $\bigcap \Sigma^+ \vdash R$ iff $\Sigma \vdash R$, for any structure R.* □

PROOF First, note the Σ has at least one prime extension, since it is consistent, by Lemma 16. Take the left-to-right case. Suppose that $\Sigma \nvdash R$. Then we can construct a prime extention Σ^+ of Σ such that $\Sigma^+ \nvdash R$, using the procedure of Lemma 15. Since $\bigcap \Sigma^+ \subseteq \Sigma^+$, we have $\bigcap \Sigma^+ \nvdash R$. Hence, by contraposition, if $\bigcap \Sigma^+ \vdash R$ then $\Sigma \vdash R$. The right-to-left case is immediate. ■

Lemma 18 *Let Σ be a consistent set of structures. If $R \notin \bigcap \Sigma^+$ then there is some prime extension Σ^+ of Σ such that $R \notin \Sigma^+$.* □

PROOF Proof is by contraposition of the following. Suppose that for every prime extension Σ^+ of Σ we have $R \in \Sigma^+$. Then $R \in \bigcap \Sigma^+$. ■

We introduce the notion of a sub-prime theory, which is essentially a consistent set of structures that is closed under deducibility. The special requirement in Definition 31 regarding disjunctive elements in standard prime theories is absent here.

Definition 32 A *sub-prime theory* is a set Σ of structures that meets the following conditions:

[21] Particular thanks to Lee Naish for the suggestion that we make use of the intersection of all prime extensions in the completeness proof, and for a conversation which lead to Lemmas 17, 19 and 20.

(i) Σ is consistent;

(ii) Σ is closed under deducibility. That is, for any R, if $\Sigma \vdash R$ then $R \in \Sigma$. □

Obviously, every prime theory is a sub-prime theory, but not vice-versa. The reason for introducing this notion is that we are unable to prove the variant of the following lemma that would state that $\bigcap \Sigma^+$ is the *least prime extension* of Σ. We can however make do with the result that $\bigcap \Sigma^+$ is less than or equal to the least prime extension(s), and that it is sub-prime. The difficulty is that in the case that $\Sigma \vdash [R; T]$, we have a guarantee that each prime extension Σ^+ of Σ contains either R or T, but not both. Some extensions may contain R and others T; some both. So we cannot show that $\bigcap \Sigma^+$ must contain either R or T to meet the disjunctive requirement of a prime theory.

Lemma 19 (Sub-prime extension lemma) *Let Σ be a consistent set of structures. $\bigcap \Sigma^+$ is a sub-prime extension of Σ.* □

PROOF $\bigcap \Sigma^+$ is a sub-prime theory since it satisfies the two criteria: (i) $\bigcap \Sigma^+$ is consistent, since there is at least one prime theory Σ^+ such that $\bigcap \Sigma^+ \subseteq \Sigma^+$, and Σ^+ is consistent, since it is a prime theory; (ii) Suppose that $\bigcap \Sigma^+ \vdash R$. Then $\Sigma \vdash R$ by Lemma 17. Hence for any Σ' such that $\Sigma \subseteq \Sigma'$, we have $\Sigma' \vdash R$ Thus for *every* prime extention Σ^+ of Σ, we have $\Sigma^+ \vdash R$, and since Σ^+ is a prime theory, $R \in \Sigma^+$. Then $R \in \bigcap \Sigma^+$; Since $\Sigma \subseteq \bigcap \Sigma^+$, $\bigcap \Sigma^+$ is an extension of Σ. ■

Lemma 20 (Least prime extensions lemma) *Let Σ be a consistent set of structures. $\bigcap \Sigma^+$ is less than or equal to the least prime extension(s) of Σ.* □

PROOF Since $\bigcap \Sigma^+$ is the intersection of all prime extensions of Σ, we have $\bigcap \Sigma^+ \subseteq \Sigma^+$ for any prime extention Σ^+ of Σ, and hence $\bigcap \Sigma^+$ is less than or equal to the least prime extensions of Σ. ■

Definition 33 First, we define the binary operation $\star : \wp(\Phi) \times \wp(\Phi) \to \wp(\Phi)$ as follows:

$$\begin{array}{rcl} \{U_1, \ldots, U_i\} \star \{V_1, \ldots, V_j\} & = & \{\, (U_1, V_1)\,, \cdots\,, (U_1, V_j)\,, \\ & & \;\;(U_2, V_1)\,, \cdots\,, (U_2, V_j)\,, \\ & & \;\;\;\;\vdots \qquad\quad \ddots \qquad\quad \vdots \\ & & \;\;(U_i, V_1)\,, \cdots\,, (U_i, V_j)\, \} \end{array}$$

This is largely, but not exclusively, for use within this definition. Then we define the binary operation $\bullet : \wp(\Phi) \times \wp(\Phi) \to \wp(\Phi)$:

$$m \bullet n = \begin{cases} m \star n & \text{if } m \star n \nvdash \bot \\ \mathsf{undefined} & \text{otherwise} \end{cases}$$

We write $m \bullet n\!\downarrow$ to state that $m \bullet n$ is defined, in uniformity with the PDM semantics. In short, $m \bullet n\!\downarrow$ iff $m \star n$ is consistent. It is necessary to introduce some level of control of inconsistency in a definition of $\bullet$, otherwise a completeness proof for a proof system for BI with $\bot$ cannot succeed.[22] The $\bullet$ operation is essentially a cartesian product, with multiplicative conjuction $(-,-)$ as the pairing operation, plus a requirement for consistency. It is important that the structures in these sets or theories be viewed modulo syntactic equivalence (or perhaps we could regard the elements of theories as equivalence classes of structures). Otherwise, to take one example, $e = \{I\}$ would fail to act as the identity element for $\bullet$. □

[22] Our definition is simpler than the following definition, adapted from Pym [2002, §5.2, p. 73] and Pym, O'Hearn & Yang [2004, p. 294], although it addresses the same problem of inconsistency:

$$\begin{array}{rcl} \{U_1,\ldots,U_i\} \bullet \{V_1,\ldots,V_j\} & = & \{\ (U_1,V_1)\ ,\ \cdots\ ,(U_1,V_j)\ , \\ & & \ \ (U_2,V_1)\ ,\ \cdots\ ,(U_2,V_j)\ , \\ & & \ \ \ \vdots \qquad \ddots \qquad \vdots \\ & & \ \ (U_i,V_1)\ ,\ \cdots\ ,(U_i,V_j)\ \}\setminus \bot(U,V) \end{array}$$

where $\bot(U,V) = \{(U_i,V_j) \mid (U_i,V_j) \vdash \bot\}$. There is no requirement in this definition that the resulting set be consistent in order that the operation to be defined. Indeed, a more obvious choice of a definedness requirement would be that the resulting set be non-empty. This construction of $\bullet$ artificially excludes individual structures which entail $\bot$. For example, each of a and $\langle a, \bot\rangle$ is a consistent structure, but $(a, \langle a, \bot\rangle)$ is inconsistent, and $(a, \langle a, \bot\rangle)$ is excluded for just that reason. Note, however, that this procedure does not guarantee that the resulting set will be consistent, just that no element taken individually entails $\bot$. Take the obvious example: $\{a, \langle a; \bot\rangle\} \bullet e = \{a, \langle a; \bot\rangle\}$ in which $\{a, \langle a; \bot\rangle\}$ is inconsistent to begin with. This example is not especially pertinent, however, since the $\bullet$ operation will usually be performed upon pairs of prime theories, which are consistent to begin with. But consider a more elaborate example, in which the seeds of inconsistency are buried more deeply. Take sets of structures m, n with $\{a, \langle b, c\rangle\} \in m$ and $\{\langle a, \langle c; \bot\rangle\rangle, b\}$. Then $\{(a, \langle a, \langle c; \bot\rangle\rangle), (\langle b, c\rangle, b)\} \subseteq m \star n$, making $m \star n$ inconsistent, although neither of these two elements taken alone entails $\bot$, and hence would not be excluded from $m \bullet n$ under Pym's definition, which makes it a rather arbitrary construction. We regard our choice of checking the consistency of the whole set as more natural (although it would require a much more expensive computation) and it certainly integrates more neatly with the PDM semantics, and simplifies the argument of our completeness proof.

Definition 34 We may construct a *canonical countermodel* as follows.[23] $\mathfrak{M}_c = \langle M, \bullet, e, \sqsubseteq, V\rangle$ is a partially-defined model in which M is the set of all prime theories (see Definition 31), together with the unit state $e = \{I\}$. We know that there exist a sufficient number of prime theories, since for each pair of structures R, T such that $R \nvdash T$, Lemma 15 guarantees the existence of at least one prime theory Σ^+ such that $R \in \Sigma^+$ and $\Sigma^+ \nvdash T$.[24] e is not a prime theory, and in that respect is a special exception in M. The *natural valuation* function $V : \mathcal{P} \to \wp(M)$ is then defined:

$$V(a) = \{m \in M \mid a \in m\}.$$

The preorder $\sqsubseteq$: $\wp(\Phi) \times \wp(\Phi) \to \mathsf{Bool}$ (where Φ is the set of all structures of BI) is defined as non-strict set inclusion,[25] that is,

$$m \sqsubseteq n \text{ iff } m \subseteq n.$$

□

Remark 11 It does not follow from $m \bullet n \downarrow$ that if $m, n \in M$, and are thus by definition prime theories, that $m \bullet n \in M$, since we have no guarantee that $m \bullet n$

[23]Suppose that we were attempting to prove completeness with respect to the new Kripke resource semantics. Then the definition of a countermodel is slightly more elaborate. A countermodel $\mathfrak{M}_c = \langle M, \bullet, e, \pi, \sqsubseteq, V\rangle$ contains in addition a special element $\pi \in M$, the inconsistent state. We define $\pi = \varnothing$, and π is plainly not a prime theory. Because π needs to be the greatest state in M under the preorder $\sqsubseteq$, we adjust the definition of $\sqsubseteq$ as follows:

$$m \sqsubseteq n = \begin{cases} \mathsf{True} & \text{if } n = \pi \\ \mathsf{False} & \text{if } m = \pi \text{ and } n \neq \pi \\ m \subseteq n & \text{otherwise} \end{cases}$$

The natural valuation function would also need adjustment:

$$V(a) = \{m \in M \mid a \in m\} \cup \{\pi\}.$$

That is, the value of an atomic structure a in the model is the set of states having a as an element, plus π. No tampering is required to handle π in the definition of the $\bullet$ operator. Note that by the definition, $\pi \bullet m = \pi$ for any m, as required. In this scheme, we would regard $m \bullet n = \varnothing = \pi$ as being defined.

[24] Indeed, it is worth noting that the same pair R, T may yield different prime theories, given different underlying enumerations of the structures of BI. It is also worth emphasising that we have no reason to think that the procedure of Lemma 15 yields *all* prime theories, and that we do not require that.

[25]It is important that $\sqsubseteq$ be defined over all theories, that is, all sets of structures m, n, and not just over the elements of M in a canonical countermodel, which except for e are prime theories. (In fact, it only needs to be defined over non-empty theories.) See our discussion in §2.7.

is a prime theory simply because m and n are. Now consider the forcing clauses for $*$ and $-\!\!*$ in the PDM semantics. The forcing clause for $*$ requires that $n \bullet n'$ be defined, that is, that it be non-empty, and that it participate in the preorder, but not that it be a element of M, and hence not that it be a prime theory. n and n' are of course required to be prime theories, by the definition of a canonical countermodel $\mathfrak{M}_c$. The forcing clause for $-\!\!*$ raises a more delicate matter. Take $\phi -\!\!* \psi$, forced at a given state $m \in M$. It states that for any prime theory $n \in M$ for which $m \bullet n$ is defined and which forces ϕ, that $m \bullet n$ is a *prime* theory in M and forces ψ. □

Proposition 5 *The binary relation $\sqsubseteq$, as defined in Definition 34, is reflexive and transitive.* □

PROOF This follows immediately from the reflexivity and transitivity of non-strict set inclusion $\subseteq$. ■

Proposition 6 *The natural valuation defined in Definition 34 satisfies the monotonicity constraint.* □

PROOF Suppose that $m \in V(a)$ and $m \sqsubseteq n$. In the case that $m = \pi$, $n = \pi$ by the definition of $\sqsubseteq$, and $\pi \in V(a)$ by the definition of V. Otherwise, $a \in m$ by the definition of the natural valuation. Then since $m \subseteq n$, we have $a \in n$, that is, $n \in V(a)$, unless $n = \pi$, in which case also $n \in V(a)$. ■

Proposition 7 *The operator $\bullet$, as defined in Definition 33, is commutative.* □

PROOF This follows from the commutativity of multiplicative conjunction modulo syntactic equivalence, and the fact that a set is unordered. ■

Proposition 8 *The definition of $\bullet$ given in Definition 33 (taken together with the definition of $\sqsubseteq$ given in Definition 34) satisfies bifunctoriality.* □

PROOF Suppose that $m \sqsubseteq n$ and $m' \sqsubseteq n'$, and $m \bullet m' \downarrow$ and $n \bullet n' \downarrow$. Now take an arbitrary R such that $R \in m \bullet m'$. Then by Definition 33 there is some pair of structures U, V such that $U \in m$ and $V \in m'$ and $(U, V) \equiv R$ under syntactic equivalence. Then $U \in n$ and $V \in n'$, and hence $(U, V) \equiv R \in n \bullet n'$. Thus $m \bullet m' \sqsubseteq n \bullet n'$. ■

Remark 12 Suppose that $m \sqsubseteq n$, and thus that $m \subseteq n$. Then, momentarily disregarding the requirement that structures be finite in size, $\bigwedge n \vdash \bigwedge m$ by weakening ($\mathsf{wl}\uparrow$). Note the reversal of positions. Also compare monotonicity, by which, if $m \Vdash R$ then $n \Vdash R$. □

Remark 13 Suppose that m is a prime theory and that $m \vdash R$, that is, that for some finite subset $\{R_1, \ldots, R_i\}$ of m, we have $\{R_1, \ldots, R_i\} \vdash R$. Now trivially, $\{R_1, \ldots, R_i\} \vdash (R_1; \ldots; R_i)$, and so $m \vdash (R_1; \ldots; R_i)$. Since m is a prime theory, we have $(R_1; \ldots; R_i) \in m$. □

Lemma 21 $\langle((R_1, T_1); \ldots; (R_i, T_i)), U\rangle \vdash \langle(R_1; \ldots; R_i), (T_1; \ldots; T_i), U\rangle$. □

PROOF We can construct the following derivation, using multiple applications of weakening $\mathsf{wl}\downarrow$ and contraction $\mathsf{cl}\downarrow$ on the left.

$$\mathsf{cl}\downarrow \frac{\mathsf{wl}\downarrow \dfrac{\langle((R_1, T_1); \ldots; (R_i, T_i)), U\rangle}{\langle(((R_1; \ldots; R_i), (T_1; \ldots; T_i)); \ldots; ((R_1; \ldots; R_n), (T_1; \ldots; T_n))), U\rangle}}{\langle((R_1; \ldots; R_i), (T_1; \ldots; T_i)), U\rangle}$$

■

We require the following primeness lemma. Cf. the statements and proofs of primeness lemmas in Routley & Meyer [1972, pp. 62f, Lemma 4] and Galmiche et al. [2005, p. 1065, Lemma 5.3]. In particular, we adapt the proof procedure of Galmiche et al. [2005] to our apparatus.

Lemma 22 (Primeness lemma) *If m is a prime theory and $n \bullet n' \subseteq m$, and n is consistent, then there is a prime theory n^+ extending n, such that $n^+ \bullet n' \subseteq m$.* □

PROOF We construct a prime theory n^+ using a variation on the procedure of Lemma 15. In the base case, $n_0 = n$. In the inductive step, construct n_{i+1} as follows, As in Lemma 15, we take an enumeration $R_1, R_2, R_3, \ldots$ of the structures of BI.

If n_i is a prime theory

$$n_{i+1} = n_i$$

otherwise, if R_i is of disjunctive form $[R_L; R_R]$

$$n_{i+1} = \begin{cases} n_i \cup \{R_L\} & \text{if } n_i \cup \{R_L\} \bullet n' \subseteq m\text{; otherwise} \\ n_i \cup \{R_R\} & \text{if } n_i \cup \{R_R\} \bullet n' \subseteq m \\ n_i & \text{otherwise} \end{cases}$$

otherwise

$$n_{i+1} = \begin{cases} n_i \cup \{R_i\} & \text{if } n_i \cup \{R_i\} \bullet n' \subseteq m \\ n_i & \text{otherwise} \end{cases}$$

and finally

$$n^+ = \bigcup_{i \geq 0} n_i$$

It is evident from the construction that $n^+ \bullet n' \subseteq m$. We can view the tests whether $n_i \cup \{R_x\} \bullet n' \subseteq m$ as proxies for the tests for the non-deducibility of T (where T is in this case unknown) in our proof of Lemma 15, and confirmation that n^+ is indeed a prime theory may be carried out similarly to the argument there. ■

Corollary 2 *If m is a prime theory and $n \bullet n' \subseteq m$, and n and n' are consistent, then there are prime theories n^+ and n'^+ extending n and n' respectively, such that $n^+ \bullet n'^+ \subseteq m$.* □

PROOF By the commutativity of $\bullet$ and two applications of Lemma 22. ■

Now we come to the main lemma of the completeness proof.

Lemma 23 (Truth lemma) *Given a canonical countermodel $\mathfrak{M}_c$ as defined in Definition 34,*

$$\mathfrak{M}_c, m \Vdash U \textit{ iff } U \in m$$

for any $m \in M$. □

PROOF We must show that $\mathfrak{M}_c, m \Vdash U$ iff $U \in m$. Note that U is never $\perp$, since every prime theory is consistent. We do, however, have to consider cases in which $\perp$ is a substructure of U. We argue by induction on the degree of U. In every sub-case of the base and inductive cases, we treat the left-to-right and then the right-to-left case. In the base case, U is either an atomic structure, or $\top$ or I.

1. (i) Suppose that $\mathfrak{M}_c, m \Vdash a$. It is immediate that $a \in m$, by the natural valuation.

   (ii) Suppose that $a \in m$. It is immediate that $\mathfrak{M}_c, m \Vdash a$ by the natural valuation.

2. (i) In the case that U is $\top$, $U \in m$ since $\top$ is an element of every prime theory, by Definition 31, given that $R \vdash \top$ for any R by the rule of inference $\mathsf{wI}\!\uparrow$.

   (ii) $\mathfrak{M}, n \Vdash \top$ for all n in any $\mathfrak{M}$, so $\mathfrak{M}_c, m \Vdash \top$ always.

3. (i) In the case that $\mathfrak{M}_c, m \Vdash I$, $e \sqsubseteq m$. Since $e = \{I\}$, we have $I \in m$.

   (ii) In the case that $I \in m$, we have $e \subseteq m$, and hence $e \sqsubseteq m$, and then $\mathfrak{M}_c, m \Vdash I$.

In the inductive step, we consider each binary connective. In each case we take the hypothesis that for any m, $\mathfrak{M}_c, m \Vdash U_L$ iff $U_L \in m$, and $\mathfrak{M}_c, m \Vdash U_R$ iff $U_R \in m$. It is necessary that we quantify universally over states m in the scope of the hypothesis, rather than restrict ourselves to the given (but of course arbitrary) m of each case, and quantify universally over the entire induction.

1. (i) Suppose that $\mathfrak{M}_c, m \Vdash (U_L; U_R)$. Then by the forcing clause for $\wedge$, we have $\mathfrak{M}_c, m \Vdash U_L$ and $\mathfrak{M}_c, m \Vdash U_R$, and so by the inductive hypothesis $U_L, U_R \in m$. Then $\{U_L, U_R\} \subset m$, and trivially $(U_L; U_R) \vdash (U_L; U_R)$. Then $(U_L; U_R) \in m$, since m is a prime theory.

   (ii) Suppose that $(U_L; U_R) \in m$. Now $(U_L; U_R) \vdash U_L$ and $(U_L; U_R) \vdash U_R$, using weakening ($\mathsf{wI}\!\uparrow$), so since m is a prime theory, $U_L \in m$ and $U_R \in m$. Then by the inductive hypothesis, $\mathfrak{M}_c, m \Vdash U_L$ and $\mathfrak{M}_c, m \Vdash U_R$, and so by the forcing clause for $\wedge$, we have $\mathfrak{M}_c, m \Vdash (U_L; U_R)$.

2. (i) Suppose that $\mathfrak{M}_c, m \Vdash [U_L; U_R]$. Then either $\mathfrak{M}_c, m \Vdash U_L$ or $\mathfrak{M}_c, m \Vdash U_R$, by the forcing clause for $\vee$. So by the inductive hypothesis we have either $U_L \in m$ or $U_R \in m$. The cut-free derivation:

$$\equiv \frac{\text{wr}\!\downarrow \dfrac{R}{[R; \bot]}}{[R; T]}$$

gives us $U_L \vdash [U_L; U_R]$ and $U_R \vdash [U_L; U_R]$. So in either case $[U_L; U_R] \in m$, since m is a prime theory.

(ii) Suppose that $[U_L; U_R] \in m$. Since m is a prime theory, either $U_L \in m$ or $U_R \in m$, by the special disjunctive condition on prime theories in Definition 31. Then by the inductive hypothesis, either $\mathfrak{M}_c, m \Vdash U_L$ or $\mathfrak{M}_c, m \Vdash U_R$, and hence by the forcing clause for $\vee$, we have $\mathfrak{M}_c, m \Vdash [U_L; U_R]$.

3. (i) Suppose that $\langle U_L; U_R\rangle \notin m$. Then since m is a prime theory, $m \nvdash \langle U_L; U_R\rangle$, and hence $m \cup \{U_L\} \nvdash U_R$ by uncurrying.[26] In the case that $m \cup \{U_L\} \vdash \bot$ we obtain $m \cup \{U_L\} \vdash U_R$ using weakening (wr↓). But this gives us a contradiction, so this case cannot apply. In the case that $m \cup \{U_L\} \nvdash \bot$, we can construct a prime theory $m' \in M$ such that $m \cup \{U_L\} \subseteq m'$ and $m' \nvdash U_R$, by the procedure of Lemma 15. $m \subseteq m'$, so we have $m \sqsubseteq m'$. Since m' is a prime theory, we have $U_R \notin m'$, and then by the inductive hypothesis, $\mathfrak{M}_c, m' \nVdash U_R$. But $U_L \in m'$, so again by the inductive hypothesis, we have $\mathfrak{M}_c, m' \Vdash U_L$. Now recall the forcing clause for $\rightarrow$. $\mathfrak{M}_c, m \Vdash \langle U_L; U_R\rangle$ iff for all n such that $m \sqsubseteq n$, we have $\mathfrak{M}_c, n \nVdash U_L$ or $\mathfrak{M}_c, n \Vdash U_R$. Hence $\mathfrak{M}_c, m \nVdash \langle U_L; U_R\rangle$. So by contraposition, if $\mathfrak{M}_c, m \Vdash \langle U_L; U_R\rangle$, then $\langle U_L; U_R\rangle \in m$. [Cf. the corresponding case in van Dalen 2004, Lemma 5.3.9, p. 170].

(ii) Suppose that $\langle U_L; U_R\rangle \in m$. Now take any $n \in M$ such that $m \sqsubseteq n$ and $U_L \in n$. Since $m \sqsubseteq n$, we have $\langle U_L; U_R\rangle \in n$. So $\{U_L, \langle U_L; U_R\rangle\} \subseteq n$. We have the derivation, using cut (ia↑):

[26] To be more rigorous, if $m \nvdash \langle U_L; U_R\rangle$, then for every finite subset $\{R_1, \ldots, R_n\}$ of m, we have $(R_1; \ldots; R_n) \nvdash \langle U_L; U_R\rangle$, and equivalently, $\nvdash \langle (R_1; \ldots; R_n); \langle U_L; U_R\rangle\rangle$. Then by syntactic equivalence, we have $\nvdash \langle (R_1; \ldots; R_n; U_L); U_R\rangle$ for any $\{R_1, \ldots, R_n\} \subseteq m$.

$$\begin{array}{r c}
\equiv & \dfrac{(U_L; \langle U_L; U_R\rangle)}{(\langle\top; U_L\rangle; \langle U_L; U_R\rangle)} \\
\mathsf{sia}\!\downarrow & \dfrac{}{\langle\langle U_L; U_L\rangle; \langle\top; U_R\rangle\rangle} \\
\mathsf{ia}\!\uparrow & \dfrac{}{\langle\top; \langle\top; U_R\rangle\rangle} \\
\equiv & \dfrac{}{U_R}
\end{array}$$

so $n \vdash U_R$. Thus by the inductive hypothesis, for any $n \in M$ such that $m \sqsubseteq n$, if $\mathfrak{M}_c, n \Vdash U_L$ then $\mathfrak{M}_c, m \Vdash U_R$. Hence $\mathfrak{M}_c, m \Vdash \langle U_L; U_R\rangle$ by the forcing clause for $\rightarrow$.

4. (i) Suppose that $\mathfrak{M}_c, m \Vdash (U_L, U_R)$. Then by the forcing clause for $*$ there exist $n, n' \in M$ such that $n \bullet n'$ is defined,[27] and $n \bullet n' \sqsubseteq m$, and $\mathfrak{M}_c, n \Vdash U_L$ and $\mathfrak{M}_c, n' \Vdash U_R$. By the inductive hypothesis, $U_L \in n$ and $U_R \in n'$, and then by Definition 33, $(U_L, U_R) \in n \bullet n'$. Then since $n \bullet n' \sqsubseteq m$, we have $(U_L, U_R) \in m$.

   (ii) Suppose that $(U_L, U_R) \in m$. $(U_L, U_R) \nvdash \bot$, since m is a prime theory. First we establish that $U_L \nvdash \bot$ and $U_R \nvdash \bot$. Consider the derivation

$$\begin{array}{r c}
\mathsf{wr}\!\downarrow & \dfrac{(R, \bot)}{(R, \langle R, \bot\rangle)} \\
\mathsf{sim}\!\downarrow & \dfrac{}{\langle\langle R, R\rangle, \bot\rangle} \\
\mathsf{im}\!\uparrow & \dfrac{}{\langle I, \bot\rangle} \\
\equiv & \dfrac{}{\bot}
\end{array}$$

   using cut ($\mathsf{im}\!\uparrow$), showing that $(R, \bot) \vdash \bot$ for any R. Obviously the same holds for $(\bot, R)$ since $(\bot, R) \equiv (R, \bot)$. We can see that if either $U_L \vdash \bot$ or $U_R \vdash \bot$, then $(U_L, U_R) \vdash \bot$, so by contraposition, we know that $U_L \nvdash \bot$ and $U_R \nvdash \bot$. Now take the consistent sets $\{U_L\}$ and $\{U_R\}$. We have $\{U_L\} \bullet \{U_R\}\!\downarrow$ and $\{U_L\}\bullet\{U_R\} = \{(U_L, U_R)\}$. Plainly $\{U_L\}\bullet\{U_R\} \sqsubseteq m$, so by two applications of the primeness lemma (Lemma 22), we have prime extensions $n \in M$ of $\{U_L\}$ and $n' \in M$ of $\{U_R\}$ such that $n \bullet n' \subseteq m$. Hence there exist $n, n' \in M$ such that $n \bullet n'\!\downarrow$ and $n \bullet n' \sqsubseteq m$ and $U_L \in n$ and $U_R \in n'$. Then by the inductive hypothesis, there exist $n, n' \in M$ such that $n \bullet n'\!\downarrow$ and $n \bullet n' \sqsubseteq m$

[27] We do not take the definedness of $n \bullet n'$ to entail that $n \bullet n'$ is a prime theory. See Remark 11. But n and n' must be prime theories, by the definition of a canonical countermodel $\mathfrak{M}_c$.

and $\mathfrak{M}_c, n \Vdash U_L$ and $\mathfrak{M}_c, n' \Vdash U_R$. Hence by the forcing clause for $*$, $\mathfrak{M}_c, m \Vdash (U_L, U_R)$.

5. (i) Before we begin this case, recall the forcing clause for $\twoheadrightarrow$. $\mathfrak{M}_c, m \Vdash \langle U_L, U_R\rangle$ iff for all $n \in M$ such that $m \bullet n$ is defined and $\mathfrak{M}_c, n \Vdash U_L$, for all $n' \in M$ such that $m \bullet n \sqsubseteq n'$, we have $\mathfrak{M}_c, n' \Vdash U_R$.

   Take the non-prime theory $\{U_L\}$ such that $U_L \in n$ and n is a prime theory. $\{U_L\}$ is consistent since n is a prime theory, and $n \vdash U_L$. Now take the set $m \bullet \{U_L\}$. (In the case that $m \bullet \{U_L\}$ is inconsistent, that is, undefined by Definition 34, $m \bullet n$ must be inconsistent for any n such that $U_L \in n$, and thus is always undefined, since $m \bullet \{U_L\} \subseteq m \bullet n$.) Suppose that $m \bullet \{U_L\} \vdash U_R$. Recall that this means that there is some finite subset of $m \bullet \{U_L\}$ from which U_R is deducible. Let $\{R_1, \ldots, R_n\}$ be a finite subset of m such that $\{(R_1, U_L), \ldots, (R_n, U_L)\}$ is finite subset of $m \bullet \{U_L\}$ and $\{(R_1, U_L), \ldots, (R_n, U_L)\} \vdash U_R$. Then $\vdash \langle((R_1, U_L); \ldots; (R_n, U_L)), U_R\rangle$. Then by Lemma 21 we have $\vdash \langle((R_1; \ldots; R_n), (U_L; \ldots; U_L)), U_R\rangle$, and then by multiple contractions ($\mathsf{cl}\downarrow$) $\vdash \langle((R_1; \ldots; R_n), U_L), U_R\rangle$, and by currying $\vdash \langle(R_1; \ldots; R_n), \langle U_L, U_R\rangle\rangle$. Then since $\{R_1, \ldots, R_n\} \subseteq m$ and m is a prime theory, $\langle U_L, U_R\rangle \in m$. Hence we have by contraposition that if $\langle U_L, U_R\rangle \notin m$ then $m \bullet \{U_L\} \nvdash U_R$, when $m \bullet \{U_L\}\downarrow$.

   Now suppose that $\langle U_L, U_R\rangle \notin m$. Then $m \bullet \{U_L\} \nvdash U_R$ when $m \bullet \{U_L\}\downarrow$, as we have just shown. By the prime extension lemma (Lemma 16, there is a prime extension $(m \bullet \{U_L\})^+$ of $m \bullet \{U_L\}$ such that $(m \bullet \{U_L\})^+ \nvdash U_R$. Then by the primeness lemma (Lemma 22), there is a prime extension n of $\{U_L\}$ such that $m \bullet n \subseteq (m \bullet \{U_L\})^+$. Hence $m \bullet n \nvdash U_R$. But we have no assurance that $m \bullet n$ is a prime theory; indeed it is most likely not. By Lemma 19, $\bigcap(m \bullet n)^+$ is the sub-prime extension of $m \bullet n$, and since $m \bullet n \nvdash U_R$, we have $\bigcap(m \bullet n)^+ \nvdash U_R$, by Lemma 17. Then since $\bigcap(m \bullet n)^+$ is a sub-prime theory, $U_R \notin \bigcap(m \bullet n)^+$ by Definition 32, and thus by Lemma 18, there exists a prime extension n' of $m \bullet n$ such that $U_R \notin n'$.

   So, to summarise our progress so far, if $\langle U_L, U_R\rangle \notin m$, either $m \bullet n$ is never defined when $U_L \in n$, or there exist an $n \in M$ such that $m \bullet n$ is defined, and a prime extension $n' \in M$ of $m \bullet n$ such that $U_L \in n$ and $U_R \notin n'$. We are now in a position to construct an ‘exception’ to the forcing clause in the case that $\langle U_L, U_R\rangle \notin m$. Take any prime theory $m \in M$ such that $\langle U_L, U_R\rangle \notin m$. Then either there exists a prime theory $n' \in M$ such that $m \bullet n\downarrow$ and $m \bullet n \subseteq n'$ and $U_L \in n$ and $U_R \notin n'$, or $m \bullet n$ is never defined when $U_L \in n$. Now by the inductive hypothesis, $U_L \in n$ iff $\mathfrak{M}_c, n \Vdash U_L$ and $U_R \in n'$ iff $\mathfrak{M}_c, n' \Vdash U_R$. So in the case that $\langle U_L, U_R\rangle \notin m$, either there exist $n, n' \in M$ such that $m \bullet n \subseteq n'$ and $\mathfrak{M}_c, n \Vdash U_L$ and $\mathfrak{M}_c, n' \nVdash U_R$, or $m \bullet n$ is never defined when $\mathfrak{M}_c, n \Vdash U_L$, and hence $\mathfrak{M}_c, m \nVdash \langle U_L, U_R\rangle$ by the forcing clause. So by contraposition, if $\mathfrak{M}_c, m \Vdash \langle U_L, U_R\rangle$, then $\langle U_L, U_R\rangle \in m$.

(ii) Suppose that $\mathfrak{M}_c, m \nVdash \langle U_L, U_R\rangle$. Then by the forcing clause for $-\!\!*$, there is some $n \in M$ such that $m \bullet n$ is defined and $\mathfrak{M}_c, n \Vdash U_L$, for which there is some $n' \in M$ such that $m \bullet n \sqsubseteq n'$ and $\mathfrak{M}_c, n' \nVdash U_R$. By the inductive hypothesis, $U_L \in n$ iff $\mathfrak{M}_c, n \Vdash U_L$ and $U_R \in n'$ iff $\mathfrak{M}_c, n' \Vdash U_R$. So since $\mathfrak{M}_c, n' \nVdash U_R$, we have $U_R \notin n'$. We can construct a derivation

$$
\begin{array}{r}
\mathsf{sim}\!\downarrow \dfrac{(U_L, \langle U_L, U_R\rangle)}{\langle\langle U_L, U_L\rangle, U_R\rangle} \\
\mathsf{im}\!\uparrow \dfrac{}{\langle I, U_R\rangle} \\
\equiv \dfrac{}{U_R}
\end{array}
\qquad \blacksquare
$$

using cut ($\mathsf{im}\uparrow$). So since n' is prime, $(\langle U_L, U_R\rangle, U_L) \notin n'$, and hence $(\langle U_L, U_R\rangle, U_L) \notin m \bullet n$. Then by Definition 33, and since by the inductive hypothesis $U_L \in n$, we have $\langle U_L, U_R\rangle \notin m$. Hence, by contraposition, if $\langle U_L, U_R\rangle \in m$ then $\mathfrak{M}_c, m \Vdash \langle U_L, U_R\rangle$.

Theorem 3 (Completeness) *If $R \vDash T$ then $R \vdash T$.* □

PROOF First we construct a canonical countermodel $\mathfrak{M}_c = \langle M, \bullet, e, \sqsubseteq, V\rangle$, as described in Definition 34. Now suppose that $R \nvdash T$. By Lemma 15, there exists a prime theory m such that $R \in m$ and $T \notin m$. By Definition 34, $m \in M$. By Lemma 23, $\mathfrak{M}_c, m \Vdash R$ and $\mathfrak{M}_c, m \nVdash T$. It follows that $R \nvDash T$, by the definition of semantic entailment (Definition 25). Then by contraposition, if $R \vDash T$ then $R \vdash T$. ■

3.5. FURTHER WORK

3.5.1. Cut elimination

Cut elimination is a highly desirable proof-theoretical property for sequent calculi, and also for systems in the calculus of structures. Here is a fairly generic example of a multiple-conclusion sequent calculus cut rule:

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \mathrm{Z} \Rightarrow \mathrm{H}}{\Gamma, \mathrm{Z} \Rightarrow \Delta, \mathrm{H}}$$

ϕ is called the *cut formula*. The cut rule in LBI is slightly more elaborate, with the cut-formula occurring at an arbitrary depth in the bunch of the right premise:

$$\frac{\Delta \Rightarrow \phi \qquad \Gamma(\phi) \Rightarrow \psi}{\Gamma(\Delta) \Rightarrow \psi}$$

The cut rule in a sequent calculus guarantees the *subformula property*. A proof has the subformula property if every formula that appears in the proof tree is a subformula of the formula which is being proven. That this is so is clear from the fact that the cut rule, generally speaking, is the only rule of inference in a system whose premises may contain a formula that need not be a subformula of some formula in the concluding sequent. A cut rule states at the level of proof that deduction is transitive, and is sometimes regarded as a kind of syntactic consistency property. In practical terms, it allows a proof to be shorter and more natural by way of a detour, and easier for a human being to find. A straightforward theorem prover based on a sequent calculus will typically search for proofs in a space of cut-free proof candidates.

Cut-elimination is usually proven in purely proof-theoretical terms, and indeed, the proof-theoretical procedure will usually provide some insight into the fine structure of the system at hand. A cut elimination theorem for a sequent calculus states that the system's cut rule is admissible in the system obtained by removing the cut rule from the system, that is, the cut-free system. That a rule is admissible in a system means that every theorem that can be proven using that system *plus* the rule can be proven using the system alone, without use of the rule. Typically, the admissibility of a rule is demonstrated by giving a procedure, or algorithm for transforming any proof in the system *plus* the rule into a proof in the bare system. It is a notable fact that although every *proof* in a sequent calculus that enjoys cut-elimination can be replaced by a cut-free proof, that this is not the case for *derivations* in general. For sequent calculi, a cut-elimination procedure is typically specified inductively, showing how each possible configuration of an application of the cut rule can be permuted a step upwards, or in the base case eliminated, in a transformed proof, with a reduction in either the cut-height (a well-founded measure of the length of the proofs of the premises of an application of cut) or cut-weight (a well-founded measure of the complexity of a cut formula) of any replacement cuts in the transformed proof. See Appendix A for my rendition of a cut-elimination proof for the standard right-sided sequent system for propositional linear logic, which I include merely as an example of the standard approach for sequent calculi.

A cut-elimination theorem for SBISg would be stated:

$$\text{If} \quad \vdash_{\mathsf{BISg}} R \quad \text{then} \quad \vdash_{\mathsf{SBISg}} R.$$

Where BISg is the down-fragment, or cut-free fragment of SBISg (see Figure 3.5). For systems in the calculus of structures, analogous methods based upon information about the permutability of rules in a system have been used [see, for instance Straßburger 2003, §9], as well as a novel method called *splitting* [Guglielmi 2004, §4]. In addition, an indirect proof-theoretical method is used by Tiu [2005, 2006][28] who proves cut-elimination for a system of intuitionistic logic in the calculus of structures by way of a correspondence between the down-fragment of that system and the cut-free sequent calculus.[29] The argument runs as follows. It is shown that if a structure R is provable in SJSg, then its counterpart $\underline{R}_J$ is provable in LJ. Then since LJ is known to enjoy cut-elimination, $\underline{R}_J$ is provable in cut-free LJ. Then it is shown that if a formula ϕ is provable in cut-free LJ that its counterpart $\underline{\phi}_S$ in the calculus of structures is provable in JSg, the down-fragment of SJSg. Hence if R is provable in SJSg, it is provable in 'cut-free'

[28] Also by Straßburger [2003, §5].

[29] Recall that our system SBISg is based upon Tiu's system SJSg in the April 2005 draft of the paper. That version of the paper also uses a standard single-conclusion sequent calculus LJ for intuitionistic logic. The LPAR 2006 version of the paper makes use of a more unusual multiple-conclusion sequent calculus LJm for intuitionistic logic.

JSg. Indeed Tiu also shows soundness and completeness by similar means, since LJ is known to be sound and complete. In this way, a cut-elimination result may be obtained without directly identifying a cut-elimination procedure in a system in the calculus of structures.

Similarly, a cut-elimination result for SBISg, supposing that it is there to be found, could be obtained by way of establishing a correspondence with the sequent calculus LBI, which is known to enjoy cut-elimination [Pym 2002, §6.2]. In our view, a cut-elimination result for SBISg achieved directly in the calculus of structures by methods of permutability or splitting remains a difficult challenge.

We have entertained the idea of another approach, by way of the semantics. The strategy would be to attempt to show that the down-fragment BISg of SBISg is complete, by way of a modification of the completeness proof that we have already presented. (The soundness of BISg follows immediately from the soundness of SBISg.) The starting point would be to change the definition of a prime theory; simply to require that a prime theory must be closed only under deducibility in BISg.

Definition 35 A *cut-free prime theory* is a set Σ of structures that meets the following conditions:

(i) $\Sigma \nvdash_{\mathsf{BISg}} \bot$;

(ii) For any R, if $\Sigma \vdash_{\mathsf{BISg}} R$ then $R \in \Sigma$;

(iii) If $[R; T] \in \Sigma$ then $R \in \Sigma$ or $T \in \Sigma$. □

But we are faced with an immediate difficulty. Derivations that require cut rules found in cBISg and not in BISg are needed at a number of critical points in the proof of the truth lemma (Lemma 23) as it presently stands. (See, notably, the right-to-left cases of the inductive step.) The completeness result we have presented is a statement about derivations, however.[30] On reflection, all that we would require for an indirect cut-elimination theorem would be a weaker completeness

[30]Refer to our earlier remarks on the use of cut in derivations in §3.4.1.

result for the cut-free system, just for theorems, and not derivations in general:[31]

$$\text{If } \ \vDash R \ \text{ then } \ \vdash_{\mathsf{BISg}} R.$$

It would seem more likely that this weaker result could be proven using a similar apparatus to that of our existing completeness proof. The cut-elimination argument would then run as follows. Suppose that $\vdash_{\mathsf{SBISg}} R$. Then by soundness, $\vDash R$, and hence, by the weak completeness of BISg, $\vdash_{\mathsf{BISg}} R$.

3.5.2. A local system

It would be desirable to find a local formulation of SBISg, along the lines of Tiu's [2005] systems SJSa and SJSp (or [2006] systems SISaq and SISp) for intuitionistic logic. Broadly speaking, a local system limits the amount of checking that needs to be performed to see whether a rule may be applied. The two sorts of checking that that are required in SBISg are (i) checking whether a context meets the polarity restriction on the application of a rule; and (ii) in the case of cut rules and down-fragment contraction rules, checking whether the two occurrences of the cut- or contraction-structure are indeed identical (or equivalent). Both of these operations incur significant computational overhead in a theorem prover implementation. In particular, SBISg allows cut- and contraction-structures to be of arbitrary weight. We may extend this idea beyond checking for the applicability of rules, to those rules (identity and down-fragment weakening) which introduce arbitrary structures into a derivation. If the weight of such arbitrary structures can be restricted, there ought to be an improvement in the performance of a proof-search implementation, since the search space of candidate proofs would be significantly reduced. So we may add a third sort of limitation imposed by a local system: (iii) limitation of the weight of arbitrarily introduced structures in identity and down-fragment weakening rules. Indeed, a limitation upon which structures may be 'deleted' by up-fragment weakening rules (and indeed by cut rules, but that case that has already been treated) is pertinent to the growth of the proof-search space. So we add: (iv) limitation of the weight of structures which may be deleted by cut

[31] Interestingly, the system **KL** of modal logic exhibits only this weak form of completeness, pertaining to theorems, but not the strong, relative form [Blackburn et al. 2001, Example 4.11].

and up-fragment weakening rules.[32]

Tiu's SJSa and SISaq impose limitations of sorts (ii), (iii) and (iv) by restricting identity, cut, weakening and contraction rules to their atomic cases, thus strongly limiting the complexity of the equivalence checking computations that are needed, and strongly curtailing the growth of the proof-search space. In a corresponding BI-system SBISa, the following rules would replace their counterparts in SBISg:[33]

$$\mathsf{aim}{\downarrow}\ \frac{S^+\{I\}}{S^+\langle a, a\rangle} \qquad \mathsf{aia}{\downarrow}\ \frac{S^+\{\top\}}{S^+\langle a; a\rangle}$$

$$\mathsf{aim}{\uparrow}\ \frac{S^-\langle a, a\rangle}{S^-\{I\}} \qquad \mathsf{aia}{\uparrow}\ \frac{S^-\langle a; a\rangle}{S^-\{\top\}}$$

$$\mathsf{acl}{\downarrow}\ \frac{S^-(a; a)}{S^-\{a\}} \qquad \mathsf{acr}{\downarrow}\ \frac{S^+[a; a]}{S^+\{a\}} \qquad \mathsf{awl}{\downarrow}\ \frac{S^-\{\top\}}{S^-\{a\}} \qquad \mathsf{awr}{\downarrow}\ \frac{S^+\{\bot\}}{S^+\{a\}}$$

$$\mathsf{acl}{\uparrow}\ \frac{S^+\{a\}}{S^+(a; a)} \qquad \mathsf{acr}{\uparrow}\ \frac{S^-\{a\}}{S^-[a; a]} \qquad \mathsf{awl}{\uparrow}\ \frac{S^+\{a\}}{S^+\{\top\}} \qquad \mathsf{awr}{\uparrow}\ \frac{S^-\{a\}}{S^-\{\bot\}}$$

These changes respect the need to maintain up-down symmetry of rules. The proof theoretical cost of these restrictions is the need to add so-called *medial* rules to the system to maintain completeness and proof-theoretical equivalence with SBISg (see Tiu [2005, 2006] for medial rules in the intuitionistic systems.) SJSa, SISaq, and SBISa are not fully local, however, because they leave untouched the checking of sort (i), that is, they still impose restrictions upon the polarity of contexts for rule application.

Tiu's SJSp builds upon SJSa (and SISp upon SISa, the propositional fragment of SISaq) to reduce the amount of polarity checking, and hence to produce a properly local system. The essential insight is that a context, or more perhaps more properly, a 'thread' of contexts, never switches polarity in the course of a derivation. This means that the polarity of each context in a candidate theorem may be calculated just once, at the outset of a proof, and that polarity labels

[32]There is a fifth sort of non-locality that does not afflict us here: some rules in some systems require that the wider context of a rule application be checked for some feature or other. Usually in sequent systems this would be a check that some non-principal formula or formulæ be of a certain sort. The usual example is the rule of promotion in linear logic $\frac{\Rightarrow ?\Gamma, \phi}{\Rightarrow ?\Gamma, !\phi}$, and similarly certain sequent rules for modal logic.

[33]a stands for any atomic structure, and the a prefix to each rule name stands for "atomic".

may then be attached to each substructure of the candidate theorem. Indeed, all structures in this system are *polarised structures*. Polarity labels are propagated throughout a derivation by modified rules of inference that check the polarity of substructures, instead of the polarity of contexts, for the applicability of rules; and which transmit the polarity labels on substructures (up or down, as you please) throughout the derivation. Following Tiu, the grammar of polarised structures S in SBISp would be given:[34]

$$\begin{array}{rcl} S & ::= & P \mid N \\ P & ::= & a^+ \mid \top^+ \mid \bot^+ \mid I^+ \mid (P;P)^+ \mid [P;P]^+ \mid \langle N;P\rangle^+ \mid (P,P)^+ \mid \langle N,P\rangle^+ \\ N & ::= & a^- \mid \top^- \mid \bot^- \mid I^- \mid (N;N)^- \mid [N;N]^- \mid \langle P;N\rangle^- \mid (N,N)^- \mid \langle P,N\rangle^- \end{array}$$

A proof of a structure R is a derivation of the unique polarised form of R from $\top^+$ or I^+.

So the work that would need to be done to obtain a local BI-system in the calculus of structures would be to identify the correct medial rules to produce the system SBISa, and then to prove the proof-theoretical equivalence of SBISa and SBISg. Then to progress to a fully local system SBISp, we would need to confirm the idea that polarity is preserved throughout any thread of contexts in the course of a derivation, by induction of the length of a derivation, and we would need to demonstrate the proof-theoretical equivalence of SBISp and SBISa.

[34] Cf. our definition of polarity for SBISg (Definition 17).

Chapter 4

Semantics for a hybrid BI

Hybrid logics are modal logics enriched with an additional class of propositional atoms called *nominals*. Nominals function as names of states of frames, which yields a great increase in the expressivity of hybrid logics over standard modal logics: it is a peculiar characteristic of standard modal logics that the basic objects of the semantics, namely states, do not have any direct counterpart in the languages. Nominal atoms may occur in compound formulæ which may or may not also contain ordinary propositional atoms. Formulæ containing only nominals, and no ordinary propositional atoms are called *pure* formulæ, and can be used to express frame properties, including some such as antisymmety which are not expressible in standard modal logics [see Blackburn et al. 2001, p. 436]. Hybrid logics usually also add a new kind of modal operator called a *satisfaction* operator (in fact, a distinct satisfaction operator for each nominal atom), and often other operators as well. Satisfaction operators give us the ability to talk about what is going on at named states in a model from a global perspective, that is, without regard to our own local position in the model. Hybrid logic first appeared in the pioneering work of Arthur Prior on temporal logic [see especially Prior 1967, chapter 5 & appendix B3]. The Sofia School [see Passy & Tinchev 1991] independently reintroduced nominals in their work on propositional dynamic logic.[1]

There has been considerable recent interest in the use of hybrid and modal in-

[1]Useful overviews of hybrid logic may be found in Blackburn et al. [2001, §7.3], Blackburn [2000b], Areces & Blackburn [2001], and Areces & ten Cate [2006]. Also see the Hybrid Logics Home Page `http://hylo.loria.fr` [Areces 2004–] for an introduction, history, bibliography and other resources.

tuitionistic logics for reasoning about resource distribution. Jia & Walker [2003, 2004] develop a modal intuitionistic logic and apply it to distributed programming, using the formulæ-as-types and proofs-as-programs paradigm. We classify their "logic of places" as a hybrid intuitionistic logic because it contains $@p$ modalities, where p denotes a *place*. A place p is not, however, a well-formed formula of their language in its own right. A formula $\phi@p$ is read as the type of a remote procedure call from the place p, yielding a value of type ϕ. More generally speaking $@p$ is a satisfaction operator, which permits us to say something about a what is the case at a named location in a model, from the point-of-view of any location in the model. Chadha, Macedonio & Sassone [2006] develop the work of Jia & Walker, formulating a Kripke semantics for an extension of their language, and proving soundness, completeness, the finite model property and decidability. They proffer an analogy of atomic formulæ with resources. In the Kripke semantics, models contain a set of states, as usual, but also a set of places for each state. Each clause of the forcing relation is parameterised over states and places. For example, the clause for satisfaction operators, adjusted to our notation, is:

$$\mathfrak{M}, m, p \Vdash \phi@q \text{ iff } q \in P_m \text{ and } \mathfrak{M}, m, q \Vdash \phi$$

where P_m is the set of places indexed to state m. Note the way in which the reach of a satisfaction statement is limited to the 'locality' of state m, which is not something we will do. Also note the way in which place symbols appear in the syntax, as indices of the satisfaction modality, and as semantic objects. The dual parameterisation over states and places, which we will henceforth call *locations*,[2] also occurs in the work that we find the most interesting, by Braüner & de Paiva [2003, 2006]. One benefit of this is that is permits the maintenance of intuitionistic monotonicity. In our proposal we will an attempt to use a simpler scheme, with a single domain of states, or resources, as is typical with classical hybrid logics. We will lose generalised monotonicity, but gain expressive power: we will be able to name resources explicitly in the formulæ of a variant of BI. This is is principal motivation of our proposal. Braüner & de Paiva take the step which is of most interest to us, and which is common to mainstream hybrid modal logics.

[2] "Place" is infelicitous in too many grammatical contexts to be a nice piece of jargon.

They introduce nominal propositional atoms into the language, and a satisfaction operator for each nominal. We regard our proposal as an experimental: a second motivation is to illustrate some of the difficulties that arise when we attempt to mix intuitionistic, modal and hybrid standpoints in the naïve way that we do, without dualising states and locations; and hence to underline the good sense behind that (less exciting) approach.

We will now give a brief account of another, directly pertinent piece of work. Biri & Galmiche [2003] develop a hybrid extension of BI called BI-Loc. They make no reference to the literature on hybrid logic. Then before we come to our own proposal, we will look at the Kripke semantics for intuitionistic hybrid logic given by Braüner & de Paiva.

4.1. BI-Loc

The language of BI-Loc is just the standard language of BI, with the addition of formulæ of the form $[l]\phi$, where $l \in \mathcal{L}$ are names of locations. The elements of $\mathcal{L}$ are not introduced as formulæ of the language in their own right. Biri & Galmiche give a Kripke semantics, a sequent calculus, soundness and completeness, and some decidability results, and study some computer science applications. They adapt, and extend the PDM semantics of BI with a modality for locations. The semantic adaptation is the introduction of the idea of a *resource tree*. Given a standard partially-defined model $\mathfrak{M} = \langle M, \bullet, e, \sqsubseteq \rangle$ and a set of location names $\mathcal{L}$, they define resource trees P recursively:

$$P ::= m \mid P|P \mid [l]P$$

where $m \in M$ and $l \in \mathcal{L}$, and in the base case, $m|m' \equiv m \bullet m'$. $|$ is commutative, associative, and has unit e; also $[l]P|[l]Q \equiv [l]P|Q$; and equivalence is transitive and congruent. An order $\preccurlyeq$ on resource trees is defined upon the basis of $\sqsubseteq$:

(i) $m \preccurlyeq Q$ iff $Q \equiv m'$ and $m \sqsubseteq m'$;

(ii) $[l]P' \preccurlyeq Q$ iff $Q \equiv [l]Q'$ and $P' \preccurlyeq Q'$;

(iii) $P'|P'' \preccurlyeq Q$ iff $Q \equiv Q'|Q''$ and $P' \preccurlyeq Q'$ and $P'' \preccurlyeq Q''$

Note that this definition is entirely responsible for the placement of located resource trees $[l]P$ in the order $\preccurlyeq$, and indeed for the entire semantic import of locations. No separate mapping to or from locations is required. Resource trees replace single states in the definition of the forcing relation; the $|$ operator replaces $\bullet$ in clauses of the forcing relation; and $\preccurlyeq$ replaces $\sqsubseteq$. $|$ comes out partially-defined, because it is constructed from a partially-defined $\bullet$. We just give the clause for location formulæ:

$$\mathfrak{M}, P \Vdash [l]\phi \text{ iff there exists } Q \text{ such that } [l]Q \preccurlyeq P \text{ and } \mathfrak{M}, Q \Vdash \phi$$

The monotonicity constraint on the valuation is internalised in the clause for atomic propositions, but that is just a matter of presentation. The requirement that $[l]Q \preccurlyeq P$ means that P must be a 'location tree' of the form $[l]P'$, and that $Q \preccurlyeq P'$, which makes the monotonicity of ϕ explicit for this clause. Otherwise, $[l]$ is a straightforward satisfaction modality.

A variant of LBI is extended with a single rule to handle locations:

$$\frac{\Gamma \Rightarrow \phi}{[l]\Gamma \Rightarrow [l]\phi}$$

Γ may not be a unit bunch, and $[l]\Gamma$ is a bunch in which every formula is of the form $[l]\phi$. The rule simply says that if a sequent is provable, any fully located form of that sequent is provable, which is not especially interesting.

4.2. BRAÜNER & DE PAIVA'S SEMANTICS FOR HYBRID INTUITIONISTIC LOGIC

This section briefly reproduces Braüner & de Paiva's Kripke semantics [2006, §3]. Their work is based upon Simpson's [1994] work on intuitionistic modal logic.[3] A formula is given by the grammar:

$$\phi ::= p \mid i \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid \bot \mid \Box\phi \mid \Diamond\phi \mid @_i\,\phi$$

[3]See Simpson [1994, §5.2] for the Kripke semantics. Simpson's work draws upon Ewald's [1986] work on intuitionistic tense logic, in which an ordered set of times is attached to each "state of knowledge".

where $p \in \mathcal{P}$ is an ordinary proposition letter, $i \in \Omega$ a nominal proposition letter. $@_i$ is a modal satisfaction operator for each i.[4] Satisfaction operators enable us to talk about what is happening at the location in a model denoted by i, from the standpoint of any other location. A model $\mathfrak{M}$ is

$$\langle M, \leq, \{D_m\}_{m\in M}, \{\sim_m\}_{m\in M}, \{R_m\}_{m\in M}, \{V_m\}_{m\in M}\rangle,$$

where M is a non-empty set of states; $\leq$ is a partial order on M;[5] D_m is a non-empty set of locations for each m, such that if $m \leq n$, then $D_m \subseteq D_n$; $\sim_m$ is an equivalence or identity relation on each D_m, such that if $m \leq n$, then $\sim_m \subseteq \sim_n$; R_m is a binary relation on D_m, such that if $m \leq n$, then $R_m \subseteq R_n$; and $V_m : \mathcal{P} \to \wp(D_m)$ is a valuation function for each m, assigning to each ordinary propositional atom p a subset of D_m, such that if $m \leq n$, then $V_m(p) \subseteq V_n(p)$. Each modal accessibility relation R_m on D_m and valuation V_m respects the equivalence relation $\sim_m$ on D_m. This is really a semantics for a class of hybrid intuitionistic modal logics; no properties such as reflexivity, transitivity, antisymmetry are stipulated of R_m, except that it conform to the equivalence relation: if $d \sim_m e$ and $d' \sim_m e'$ and dR_me then $d'R_me'$. Note the monotonicity conditions on each of D_m, $\sim_m$, R_m and V_m, which correspond to the familiar intuitionistic monotonicity on valuations in Kripke resource semantics. Plainly, the same location may occur in D for different states m. We can say that they are 'trans-state' locations.

There is one more ingredient before we get to the forcing relation. The function $g : \Omega \to \bigcup_{m\in M} D_m$ assigns a location somewhere in the model to each nominal propositional atom. Then the forcing relation is defined:

[4] We have changed Braüner & de Paiva's notation $i : \phi$ for the satisfaction operator to $@_i \phi$ for uniformity with the rest of our presentation, although their notation emphasises the view of satisfaction operators as labels.

[5] A preorder $\sqsubseteq$ ought to suffice.

$\mathfrak{M}, g, m, d \Vdash p$ iff $d \in V_m(p)$
$\mathfrak{M}, g, m, d \Vdash i$ iff $d \sim_m g(i)$
$\mathfrak{M}, g, m, d \Vdash \bot$ never
$\mathfrak{M}, g, m, d \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, g, m, d \Vdash \phi$ and $\mathfrak{M}, g, m, d \Vdash \psi$
$\mathfrak{M}, g, m, d \Vdash \phi \vee \psi$ iff $\mathfrak{M}, g, m, d \Vdash \phi$ or $\mathfrak{M}, g, m, d \Vdash \psi$
$\mathfrak{M}, g, m, d \Vdash \phi \to \psi$ iff for all $n \in M$ such that $m \leq n$, $\mathfrak{M}, g, n, d \nVdash \phi$ or $\mathfrak{M}, g, n, d \Vdash \psi$
$\mathfrak{M}, g, m, d \Vdash \Box\phi$ iff for all $n \in M$ such that $m \leq n$, for all $d' \in D_n$, dR_nd' implies $\mathfrak{M}, g, m, d' \Vdash \phi$
$\mathfrak{M}, g, m, d \Vdash \Diamond\phi$ iff for some $d' \in D_m$, dR_md' and $\mathfrak{M}, g, m, d' \Vdash \phi$
$\mathfrak{M}, g, m, d \Vdash @_i\phi$ iff $\mathfrak{M}, g, m, g(i) \Vdash \phi$

The first thing to note is that this semantics is divided into orthogonal intuitionistic and modal dimensions. The modal accessibility relation for each m is distinct from the intuitionistic ordering, although it 'grows' monotonically as we move up the intuitionistic order. A standard generalised monotonicity result holds over the intuitionistic order. There is no duality of $\Box$ and $\Diamond$ operators, as in classical modal logics. We note the essential hybrid elements: a nominal atom is assigned to, or names, exactly one location, and the satisfaction operator $@_i$ for each nominal atom i allows us to talk about what is the case at $g(i)$, regardless of our current location, although only within the confines of the current intuitionistic state. Satisfaction operators only free us from our modal position.

Braüner & de Paiva devote a good part of their attention to the development of a natural deduction system for intuitionistic modal logic. The proof-theory of hybrid logic is typically done with labelled systems, in which every formula of a proof has a label attached. Labels correlate with locations. In Braüner & de Paiva's system, labelling is internalised in the language, by use of satisfaction operators. The rules of inference are all for satisfaction formulæ. A number of proof-theoretical presentations of intuitionistic modal logic [e.g. Simpson 1994] make use of labelled systems, in which formulæ of the language are labelled. Typically, to prove a sequent, you would begin by labelling each of its formulæ with the same arbitrary label, one not contained in any labelled formula in the sequent.[6] With hybrid logics, this labelling can be internalised in an obvious way:

[6]For other work on the development of proof theory for hybrid logics, see Seligman [1991,

a labelled formula $i : \phi$ may simply be seen as an unlabelled formula $@_i\,\phi$ of the language. The 'situated' formula ϕ can simply be seen as being modified by a satisfaction operator. As we noted, Braüner & de Paiva would actually write the formula $@_i\,\phi$ as $i : \phi$, with "i:" read as a modal satisfaction operator, not a label, although it makes their system look like a more conventional labelled system.

4.3. HBI

We now present our own preliminary proposal of a logic HBI, which is an extension of BI in the style of a hybrid modal logic.[7] The treatment is rudimentary, and entirely from a semantic point of view. We give a Kripke resource semantics for HBI, but no proof-theoretical treatment. Indeed, we do not know whether there exists a reasonable proof theory for HBI.

We wish to avoid the introduction of an additional semantic index, like location. As we have seen, it is typical in the intuitionistic hybrid logics in the literature for the forcing relation to be parameterised over two domains of objects in a

1997, 2001], treated further by Blackburn [2000a]; and Braüner [2004a, 2004b]. There is also work on display calculi for nominal tense logic by Demri & Goré [2002].

[7]Initially, we examined the possibility of characterising BI by an embedding into an extension of the modal logic S4, in a way analogous to a standard embedding of intuitionistic logic into S4. One such embedding [Troelstra & Schwichtenberg 2000, §9.2] is:

$$
\begin{array}{rcl}
p^{\Box} & := & \Box p \\
\bot^{\Box} & := & \bot \\
(\phi \wedge \psi)^{\Box} & := & \phi^{\Box} \wedge \psi^{\Box} \\
(\phi \vee \psi)^{\Box} & := & \phi^{\Box} \vee \psi^{\Box} \\
(\phi \rightarrow \psi)^{\Box} & := & \Box(\phi^{\Box} \rightarrow \psi^{\Box})
\end{array}
$$

The first embedding of this sort was found by Gödel [1933]. Observe that in this case the embedding of a proposition letter p as $\Box p$ does the work of the monotonicity constraint in the semantics of intuitionistic logic and BI. Such a constraint does not appear in the Kripke semantics for S4 or for modal logics generally. Something similar applies for the conditional; compare the forcing clauses for the intuitionistic conditional to the classical conditional of modal logics, which considers only the current state of a model. So the idea here would be that the work of the monotonicity constraint would be done in the embedding translation, but that no such constraint would be placed upon the underlying valuation function. Hence nominals might be allowed to evade the monotonicity constraint simply by embedding i as i and not as $\Box i$. Also note that it is possible to construe $*$ as a standard two-place modality if an additional binary relation, the reverse of the preorder, is introduced.

model: the usual states, and a separate domain of objects named by the nominals. In this sort of arrangement, nominals cannot name resources, as we understand resources in BI. The major drawback will be that the usual monotonicity constraint cannot, by definition, be satisfied by the valuation function over nominals. In fact, Braüner & de Paiva mention the possibility of such a hybridisation of intuitionistic logic "as a language for talking about intuitionistic Kripke structures", but warn that "[c]hoosing this option puts an excessive emphasis on Kripke semantics as a guiding principle." [2006, p. 236].

In addition to the usual propositional letters, HBI contains an additional class of propositional letters, called *nominals*, which denote states of models. Braüner & de Paiva [2006, §1] point out that nominals play a similar rôle in hybrid logics to that played by constants (or names) in first-order logic. Under a resource interpretation of HBI, nominals name resources. HBI also contains a satisfaction modality $@_i$ for each nominal i. A stronger extension might include a bind operator $\downarrow$ for what is essentially existential quantification over nominals. We formulate the semantics with a single domain of states in a model, that is, without the duality of states and locations.

Propositions of BI admit of a declarative reading, as statements *about* resources or 'resource situations'.[8] But crucially, a proposition of BI is not itself seen as a resource. (This is to be contrasted with the standard view in linear logic of propositions as resources and proofs as actions upon resources; a view that is proof-theoretically motivated.) In fact, resources that may be part of the subject matter of a statement go without explicit mention. We can say in this sense that propositions of BI lack referential transparency. Consider the proposition $coin \mathrel{-\!\!*} choc$. We can read this as: "If I had another coin, I could buy a chocolate". If we follow closely the forcing clause for $\mathrel{-\!\!*}$, we can give the following gloss: if the present 'resource situation' were 'combined' with any resource situation in which I had a another coin[9] the combination would be a resource situation in

[8] See Pym's [2002, pp. xxxii–xxxix] discussion, which contains the *coin* and *choc* example, and the comparison with linear logic. "Declarative" is Pym's adjective. Particularly helpful is the comparison of $\mathrel{-\!\!*}$ with linear implication (recall the well-known embedding of intuitionistic implication $\phi \rightarrow \psi$ as $!\phi \multimap \psi$) which illustrates the failure of a 'use counting' approach to multiplicative implication in BI.

[9] We emphasise, *another* coin. It is sometimes natural to restrict ourselves to models in which $\bullet$ is a combination of *disjoint* resources.

which I could buy a chocolate. Of course, the chocolate might cost more than one coin, but we are talking about a 'resource shortfall' of one coin. On the other hand, we read the statement *coin* $\multimap$ *choc* of linear logic as stating that I can expend exactly one coin to obtain a chocolate. With linear logic, the resource is the coin (and the chocolate of course), and there is no background resource situation to consider. In BI we can make a statement that certain additional resources are needed to reach a given outcome, without any explicit mention of the resources which are already available. Take as an example of a background resource a global variable in an imperative computer program, which although not passed explicitly as an argument to a function, and not figuring in any type signature for the function, may nonetheless be read or written by the function.

Our suggestion is that HBI might allow a mixed approach to reasoning about resources, where we have these sorts of declarative statements intermingled with denotational statements about resources, such that we can reason about explicitly named resources. Although resources do not appear directly in the syntax of standard BI, they are the building-blocks of Kripke resource semantics: the states, or possible worlds of the models are be regarded as resources, and the operation $\bullet$ combines resources in some underdetermined way.

We introduce nominal atoms as the names of resources, that is, as names of states or possible worlds. We enrich the usual language of BI with nominal proposition letters $i, j, k, \ldots \in \Omega$, and for each of these, a *satisfaction* operator $@_i, @_j, @_k, \ldots$. There is a special nominal $o \in \Omega$ which denotes the unit state e. We also introduce a modal $\Diamond$ operator. Thus we have this grammar for well-formed formulæ of HBI:

$$\phi ::= p \mid i \mid o \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid I \mid \phi * \phi \mid \phi \mathrel{-\!\!*} \phi \mid \Diamond\phi \mid @_i\,\phi$$

The set of nominal proposition letters, or nominal atoms Ω is enumerable, and is disjoint from the set of *ordinary* propositional letters $\mathcal{P}$. The set $\mathcal{P} \cup \Omega$ is called the set of *atoms*. We propose an extension of the standard PDM semantics for BI.[10] A frame $\mathfrak{F} = \langle M, \bullet, e, \sqsubseteq, \sim \rangle$ is a standard frame for BI enriched with an identity

[10] See Galmiche et al. [2005, §5.3] and our §2.7.

relation $\sim$ on M. $\sim$ must satisfy the constraint that $m \sim n$ only if $m \sqsubseteq n$ and $m \sqsupseteq n$. We do not require that if $m \sqsubseteq n$ and $m \sqsupseteq n$ then $m \sim n$. That is, we do not require that $\sqsubseteq$ be antisymmetric. A model $\mathfrak{M} = \langle \mathfrak{F}, V, g \rangle$ is a frame together with a valuation function $V : \mathcal{P} \rightarrow \wp(M)$ assigning ordinary proposition letters to sets of states, and an *assignment* function $g : \Omega \rightarrow M$ assigning a single state to each nominal proposition letter. Every assignment g must satisfy the constraint that $g(o) = e$. Thus nominals can be regarded as names for states, and of course, a state may have more than one name, or no name at all. Any valuation function V on the ordinary propositions must satisfy the usual intuitionistic monotonicity constraint:

$$\text{if } m \in V(p) \text{ and } m \sqsubseteq n, \text{ then } n \in V(p)$$

We propose the following definition of the forcing relation:

$\mathfrak{M}, m \Vdash p$ iff $m \in V(p)$
$\mathfrak{M}, m \Vdash i$ iff $m \sim g(i)$
$\mathfrak{M}, m \Vdash \top$ always
$\mathfrak{M}, m \Vdash \bot$ never
$\mathfrak{M}, m \Vdash \phi \wedge \psi$ iff $\mathfrak{M}, m \Vdash \phi$ and $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \vee \psi$ iff $\mathfrak{M}, m \Vdash \phi$ or $\mathfrak{M}, m \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \rightarrow \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$,
  $\mathfrak{M}, n \nVdash \phi$ or $\mathfrak{M}, n \Vdash \psi$
$\mathfrak{M}, m \Vdash I$ iff $e \sqsubseteq m$
$\mathfrak{M}, m \Vdash \phi * \psi$ iff for some $n, n' \in M$ such that $n \bullet n' \downarrow$ and $n \bullet n' \sqsubseteq m$,
  $\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$
$\mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi$ iff for all $n \in M$ such that $m \bullet n \downarrow$ and $\mathfrak{M}, n \Vdash \phi$,
  $\mathfrak{M}, m \bullet n \Vdash \psi$
$\mathfrak{M}, m \Vdash \Diamond \phi$ iff for some n such that $m \sqsubseteq n$, $\mathfrak{M}, n \Vdash \phi$
$\mathfrak{M}, m \Vdash @_i \phi$ iff $\mathfrak{M}, g(i) \Vdash \phi$

The forcing clauses of the standard PDM semantics are unaltered. It might be tidier now that we are dealing with the two valuations V, g to drop the monotonicity constraint on V and internalise it in the forcing relation, thus:

$$\mathfrak{M}, m \Vdash p \text{ iff for some } n \text{ such that } n \sqsubseteq m,\ n \in V(p)$$

but we will keep to our usual practice. In addition, we can could consider extending the language with a bind operator $\downarrow$, having the following forcing clause:[11]

$$\mathfrak{M}, m \Vdash {\downarrow} x.\phi \text{ iff for some } i \in \Omega, \mathfrak{M}, m \Vdash \phi[i/x]$$

where $\phi[i/x]$ denotes the formula obtained by simultaneously replacing all unbound occurrences of x with i.

Regardless of which state m we are at, $@_i\,\phi$ is true iff ϕ is true at the state named by i. A satisfaction operator makes you forget where you are, and consider the situation elsewhere. If a satisfaction formula is satisfied in a model, it is valid in that model. We can then express, for example, the fact that i and j name the same resource by writing $@_i\, j$. By writing $i \wedge \phi$, we say that the statement ϕ is true of the resource denoted by i *and* that we are situated at the state named by i. Formulæ of these forms allow us to talk about the attributes of particular, named resources in a model, from global and situated perspectives. Nominals occurring in a formula carry the 'force of circumstance'. o is the name of e, and the situated counterpart of the propositional constant I: it is the least state at which I holds. With the inclusion of o, the formula $o \to I$ is e-valid.

The $\Diamond$ resembles the $\Diamond$ of S4 because $\sqsubseteq$ is a reflexive and transitive. In general, we can express in formulæ the ordering of states denoted by nominals:

$$@_i\, \Diamond j \text{ iff } g(i) \sqsubseteq g(j).$$

This statement expresses a fact about the frame's preorder from a global perspective, asserting nothing about the situation in which it is uttered. The related, situated statement $i \wedge \Diamond j$ holds iff $g(i) \sqsubseteq g(j)$ *and* it is uttered in the situation of $g(i)$. Under the resource interpretation we can read $\Diamond i$ as "$g(i)$ would be a sufficient substitute for the current resource".

It can be seen immediately that generalised monotonicity does not hold for this semantics. That is, it does not follow from $\mathfrak{M}, m \Vdash \phi$ and $m \sqsubseteq n$ that $\mathfrak{M}, n \Vdash \phi$ for an arbitrary formula ϕ. This is precisely because the assignment function g does not obey the monotonicity constraint: a nominal holds at exactly one state. If a nominal holds at a given state, it cannot hold at any other state, and hence not

[11] Again we have an clash of standard notation, but $\downarrow$ is conventionally used as the bind operator in hybrid logics.

at any distinct state that is placed equally or higher in the preorder. Generalised monotonicity does hold, however, for formulæ containing no free nominals, that is, no nominals outside the scope of any satisfaction operator, as will show in the next section.

The statement $\top \to \phi$ means the same as $\Box\phi$ in S4. Suppose we added a $\Box$ operator, having the forcing clause:

$$\mathfrak{M}, m \Vdash \Box\phi \text{ iff for all } n \text{ such that } m \sqsubseteq n,\ \mathfrak{M}, n \Vdash \phi$$

Then we could easily see that $\mathfrak{M}, m \Vdash \Box\phi$ iff $\mathfrak{M}, m \Vdash \top \to \phi$. HBI, like standard BI and intuitionistic logic, has the expressive power of $\Box$ in S4 built-in to the forcing clause for $\to$, because the meaning of $\to$ is monotonic and the relation $\sqsubseteq$ is reflexive and transitive.[12] It is just that those other systems are better-behaved; by maintaining generalised monotonicity, they give us $\phi \vDash \top \to \phi$ and $\top \to \phi \vDash \phi$. But although in HBI, we do not get the mutual semantic entailment of ϕ and $\top \to \phi$, it is still the case that $\mathfrak{M} \Vdash \phi$ iff $\mathfrak{M} \Vdash \top \to \phi$. That is, they are equivalent for validity (and indeed e-validity) in a model. So the proof-theoretic equivalence of ϕ and $\top \to \phi$ is not in such danger after all; it should certainly obtain for (alleged) theorems and their sub-formulæ, or more generally sub-formulæ in sequents. We can say that $\varnothing_a \Rightarrow \phi$ iff $\varnothing_a \Rightarrow \top \to \phi$, but this is not to say that the statements ϕ and $\top \to \phi$ *mean* the same thing. They are semantically distinct in the absence of generalised monotonicity.[13]

Let us consider some more examples of the expressive power of HBI. A formula $i * j$ says that the present state or resource is higher in the preorder than, or equally placed with, the combination of the two resources named by i and j.

[12] Standard BI does not contain the expressive power of $\Diamond$ in S4, though. Like intuitionistic modal logics generally, HBI lacks the duality of boxes and diamonds that we find in classical modal logics. We cannot show by a semantic argument that $\mathfrak{M}, m \Vdash \top \to \phi$ iff $\mathfrak{M}, m \Vdash \Diamond(\phi \to \bot) \to \bot$. But we can obtain a neat characterisation of intuitionistic double negation: $\mathfrak{M}, m \Vdash (\phi \to \bot) \to \bot$ iff $\mathfrak{M}, m \Vdash \Box\Diamond\phi$. We prove this as follows. $\mathfrak{M}, m \Vdash (\phi \to \bot) \to \bot$ *iff* there is no n such that $m \sqsubseteq n$ and $\mathfrak{M}, n \Vdash \phi \to \bot$ *iff* for every n such that $m \sqsubseteq n$ there is some n' such that $n \sqsubseteq n'$ and $\mathfrak{M}, n' \Vdash \phi$ *iff* $\mathfrak{M}, m \Vdash \Box\Diamond\phi$.

[13] Cf. the use of the generalisation rule in Hilbert-style proof systems for modal logic. Although there is no axiom $p \to \Box p$, there is a rule of inference $\dfrac{\vdash \phi}{\vdash \Box\phi}$ which acts upon theorems, but not upon formulæ in general.

Remember that we read $\sqsubseteq$ as a comparison of the 'sufficiency' of resources.[14] So on this reading, $i * j$ says that the present resource[15] is sufficient for any task that requires the $\bullet$-combination of the particular resources named by i and j. A formula $i \mathrel{-\!*} \phi$ says that either ϕ holds for the $\bullet$-combination of the resource named by i with the present resource, or that that combination is undefined. $i \to \phi$ says that if the resource named by i is greater than or equal to the present resource, then ϕ holds for the resource named by i. Although it is monotonic in its ambit, it really only says something about at most one resource. At most one because our commitment to the claim about $g(i)$ is conditional upon the modal accessibility of $g(i)$ from the present state. This differs from the meaning of $@_i\,\phi$, which says unconditionally, or globally, that ϕ holds at the resource named at i. $i \to \phi$ holds in the case that $g(i)$ is not accessible. $i \vee \phi$ says that either ϕ holds for the present resource, or the present resource is the one named by i. $i \vee j$ says that the present resource is the one named by i or the one named by j. Conditionals with nominal consequents are peculiar in their meaning. $\phi \to i$ says that ϕ holds for no resource greater than or equal to the present one, with the exception of $g(i)$, if indeed $g(i)$ is greater than or equal to the present resource. It can be thought of as an 'almost-negation'. $\phi \mathrel{-\!*} i$ says that whenever ϕ holds for a resource, and the $\bullet$-combination of that resource with the present resource is defined, that that combination is the resource named by i. Consequently, if the combination is defined, ϕ holds at exactly one resource in the model (and furthermore, ϕ must be a nominal, or a loose formula, as defined in the following section, or hold only at an upper bound of $\sqsubseteq$). For any of these examples in which we have used the form of words 'for the present resource', we can produce, by the application of a $@_j$ operator, a global or delocated example in which "the present resource" is replaced by the resource $g(j)$ which is named by j.

There is a limited sense in which we can capture the content of classical implication using nominals and satisfaction operators. A forcing clause for classical

[14] The semantics of BI are not sophisticated enough to express comparisons of resources with respect to sufficiency *for a given task*. It does, however, permit us to express the idea that any resource that is sufficient for a task represented by the formula ϕ is sufficient to carry out the task represented by the formula ψ, by writing $\phi \to \psi$.

[15] By which I mean the state m of the model at which we presently stand. We might talk instead about the resources available to us in our present situation.

implication will refer only to the current state. We can simulate a classical conditional $\phi \to \psi$, at the cost of anonymity, using the 'narrowing' effect of $@_i$:

$$i \wedge (@_i\, \phi \to @_i\, \psi)$$

Wherever we are in the model, the truth of $@_i\, \phi \to @_i\, \psi$ depends only on the situation at i. Then the conjunction with i asserts circumstantially that we are at $g(i)$. This formula does not express classical implication in a 'portable' way, though, because it is bound to i. A variation on this idea is:

$$i \wedge \phi \to \psi$$

which says that if $g(i)$ is accessible and ϕ holds there, then ψ holds there.

Blackburn et al. [2001, p. 438f] give axioms for normal hybrid logics. The axioms involving satisfaction operators, with the exception of self-duality, are valid in HBI, even for intuitionistic implication. We just mention these. i stands for an arbitrary nominal and ϕ for any formula.

$@_i\,(\phi \to \psi) \to (@_i\, \phi \to @_i\, \psi)$	$\mathbf{K}_@$
$i \wedge \phi \to @_i\, \phi$	INTRODUCTION
$@_i\, i$	REF
$@_i\, j \leftrightarrow @_j\, i$	SYM
$@_i\, j \wedge @_j\, \phi \to @_i\, \phi$	NOM
$@_j\, @_i\, \phi \leftrightarrow @_i\, \phi$	AGREE
$\Diamond @_i\, \phi \to @_i\, \phi$	BACK

The formula $\Diamond i \wedge @_i\, \phi \to \Diamond \phi$ (BRIDGE) is readily seen to be valid, as is $(i \wedge @_i\, \phi) \to \phi$ (ELIMINATION), although it is obtained from INTRODUCTION in the classical system by self-duality. $\mathbf{K}_@$ is a variation on the standard axiom for modalities, and holds in our intuitionistic context. REF, SYM, NOM, AGREE express the properties of names. AGREE says that only the innermost satisfaction operator matters. NOM gives us transitivity of names.

4.4. LOOSE FORMULÆ AND MONOTONICITY

We conclude with a result admitting a limited form of monotonicity to HBI.

Definition 36 A formula is *loose* if it has the following recursively defined syntactic form:

(i) Any nominal atom is loose;

(ii) $\phi \wedge \psi$ is loose iff either ϕ is loose or ψ is loose;

(iii) $\phi \vee \psi$ is loose iff either ϕ is loose or ψ is loose;

(iv) $\phi \mathrel{-\!\!*} \psi$ is loose iff ψ is loose;

(v) A formula of any other form is not loose.

Any formula which is not loose is *tight*. □

All formulæ of the forms $\phi \to \psi$, $\phi * \psi$ and $@_i \phi$ are tight, as are ordinary propositional atoms and the logical constants $\top$, $\bot$ and I.

Theorem 4 (Qualified, or Tight Monotonicity)

$$\text{If } \mathfrak{M}, m \Vdash \phi \text{ and } \phi \text{ is tight and } m \sqsubseteq n \text{ then } \mathfrak{M}, n \Vdash \phi$$

□

PROOF The proof is by induction on the depth of a formula. First, the base cases.

1. If $\mathfrak{M}, m \Vdash p$ then $\mathfrak{M}, n \Vdash p$ for all n such that $m \sqsubseteq n$, by the monotonicity constraint on the valuation function V;

2. i is not tight;

3. $\mathfrak{M}, n \Vdash \top$ for all n;

4. It is never the case that $\mathfrak{M}, m \Vdash \bot$.

5. If $\mathfrak{M}, m \Vdash I$ then $e \sqsubseteq m$, so for all n such that $m \sqsubseteq n$, $e \sqsubseteq n$ by the transitivity of $\sqsubseteq$, so $\mathfrak{M}, n \Vdash I$.

Now, the inductive cases. The inductive hypothesis is that if ϕ is tight and $\mathfrak{M}, m \Vdash \phi$, then $\mathfrak{M}, n \Vdash \phi$ for all n such that $m \sqsubseteq n$.

1. Suppose that $\phi \wedge \psi$ is tight. Then ϕ and ψ are both tight. Now suppose that $\mathfrak{M}, m \Vdash \phi \wedge \phi$. Then $\mathfrak{M}, m \Vdash \psi$ and $\mathfrak{M}, m \Vdash \psi$, by the forcing clause for $\wedge$. Then by the inductive hypothesis (twice, for ϕ and for ψ), $\mathfrak{M}, n \Vdash \psi$ and $\mathfrak{M}, n \Vdash \phi$, for all n such that and $m \sqsubseteq n$. Then by the forcing clause for $\wedge$, $\mathfrak{M}, n \Vdash \psi \wedge \phi$ for all n such that and $m \sqsubseteq n$.

2. Suppose that $\phi \vee \psi$ is tight. Then ϕ and ψ are both tight. Now suppose that $\mathfrak{M}, m \Vdash \phi \vee \phi$. Then $\mathfrak{M}, m \Vdash \psi$ or $\mathfrak{M}, m \Vdash \psi$, by the forcing clause for $\vee$. Then by the inductive hypothesis, in the case that $\mathfrak{M}, m \Vdash \psi$, $\mathfrak{M}, n \Vdash \psi$ for all n such that and $m \sqsubseteq n$, and in the case that $\mathfrak{M}, m \Vdash \phi$, $\mathfrak{M}, n \Vdash \phi$ for all n such that and $m \sqsubseteq n$. Then by the forcing clause for $\vee$, in either case, $\mathfrak{M}, n \Vdash \psi \vee \phi$ for all n such that and $m \sqsubseteq n$.

3. $\phi \to \psi$ is always tight. The following holds regardless of whether either ϕ or ψ is loose or tight. The inductive hypothesis is not required, and in this sense, monotonicity is built into the forcing clause for $\to$. Suppose that $\mathfrak{M}, m \Vdash \phi \to \psi$. Then for any n such that $m \sqsubseteq n$, either $\mathfrak{M}, n \nVdash \phi$ or $\mathfrak{M}, n \Vdash \psi$. For any n' such that $n \sqsubseteq n'$, we have $m \sqsubseteq n'$ by the transitivity of $\sqsubseteq$, so for any n' such that $n \sqsubseteq n'$, we have either $\mathfrak{M}, n' \nVdash \phi$ or $\mathfrak{M}, n' \Vdash \psi$. Hence for any n such that $m \sqsubseteq n$, $\mathfrak{M}, n \Vdash \phi \to \phi$, by the forcing clause for $\to$.

4. $\phi * \psi$ is always tight. Suppose that $\mathfrak{M}, m \Vdash \phi * \psi$. Then there exist n, n' such that $n \bullet n'$ is defined and $n \bullet n' \sqsubseteq m$ and $\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$. Then for any m' such that $m \sqsubseteq m'$, there exist the same n, n' such that $n \bullet n' \sqsubseteq m'$ and $\mathfrak{M}, n \Vdash \phi$ and $\mathfrak{M}, n' \Vdash \psi$, by the transitivity of $\sqsubseteq$. Again, the inductive hypothesis is not required.

5. Suppose that $\phi \mathrel{-\!\!*} \psi$ is tight. Then ψ is tight. Now suppose that $\mathfrak{M}, m \Vdash \phi \mathrel{-\!\!*} \psi$. Then for any n such that $\mathfrak{M}, n \Vdash \phi$ and $m \bullet n$ is defined, we have $\mathfrak{M}, m \bullet n \Vdash \psi$. Now suppose that $m \sqsubseteq m'$. By the bifunctoriality of $\bullet$, $m \bullet n \sqsubseteq m' \bullet n$, when $m \bullet n$ and $m' \bullet n$ are defined. Then in the case that ψ is tight, we have by the inductive hypothesis that for any n such that $\mathfrak{M}, n \Vdash \phi$ and $m' \bullet n$ is defined, $\mathfrak{M}, m' \bullet n \Vdash \psi$, and hence that $\mathfrak{M}, m' \Vdash \phi \mathrel{-\!\!*} \psi$ by the forcing clause for $\mathrel{-\!\!*}$. ■

Generally speaking, loose formulæ are non-monotonic. An additive conjunction where one conjunct is loose holds at at most one state. It does not exhibit monotonic behaviour. (Unless, trivially, that state is an upper bound of $\sqsubseteq$.) An additive disjunction where one conjunct is loose may hold at states which are not comparable using $\sqsubseteq$, or between which a gap lies, that is, given an ordering $n \sqsubseteq n' \sqsubseteq n''$, it may hold at n and n'', but not n'.

It might be a reasonable suggestion that since we have relinquished full generalised monotonicity, that it would not be such a bad thing to modify the definition of the forcing relation in ways that would weaken the tight monotonicity result of Theorem 4, but increase the expressivity of HBI. We could remove explicit monotonicity from the forcing clause for $*$, thus:

$$\begin{aligned}&\mathfrak{M}, m \Vdash \phi * \psi \text{ iff for some } n, n' \in M \text{ such that } n \bullet n' \sim m,\\ &\qquad \mathfrak{M}, n \Vdash \phi \text{ and } \mathfrak{M}, n' \Vdash \psi\end{aligned}$$

Consider that in the above proof, and well as in the original proof of generalised monotonicity (Lemma 1), that the inductive hypothesis is not required in the case for $*$, which is what allows all formulæ $\phi * \psi$ to be tight. With this modification, a formula $i * j$ would have a more particular meaning: that the present state is the $\bullet$-combination of the states named by i and j. This would allow us to use $*$ to construct names for compound resources, which would seem most desirable.

We might imagine that the cost would be that formulæ of the form $\phi * \psi$ could only be classified as tight if ϕ and ψ were each tight. But even then, the case for $*$ in the proof of a modified tight monotonicity result would falter as follows: Suppose that $\phi * \psi$ is tight. Then ϕ and ψ are tight. Now suppose that $\mathfrak{M}, m \Vdash \phi * \psi$. Then there exist n', n'' such that $n' \bullet n'' \sim m$ and $\mathfrak{M}, n' \Vdash \phi$ and $\mathfrak{M}, n'' \Vdash \psi$. By the inductive hypothesis, $\mathfrak{M}, m' \Vdash \phi$ for all m' such that $n' \sqsubseteq m'$ and $\mathfrak{M}, m'' \Vdash \psi$ for all m'' such that $n'' \sqsubseteq m''$. By bifunctoriality, $n' \bullet n'' \sqsubseteq m' \bullet m''$ for any such m', m''. So we have $\mathfrak{M}, m' \bullet m'' \Vdash \phi * \psi$ for any m', m'' such that $m \sqsubseteq m' \bullet m''$. *But this does not put us in position to say anything* about n such that $m \sqsubseteq n$ in general; there may exist n such that $m \sqsubseteq n$, and $n \nsim m' \bullet m''$ for any of these m', m''. So it looks as if $\phi * \psi$ would have to be classified as loose, even when ϕ and ψ are tight. Moreover, we can see that if this change were make to any of the Kripke resource semantics for standard BI, that generalised monotonicity would fail.

Conclusion

We hope that we have illuminated some of the interstices of the logic of bunched implications, in particular of its Kripke resource semantics, and the interaction of the semantics and proof theory of BI. We hope too to have indicated how BI may be understood as a logic of resources, looking through a semantic lens, and especially through our exploration of various hybrid semantics, including our own tentative semantics for HBI, which represents an attempt to introduce *names for resources* into BI. We have tried to show how the resource view of BI is motivated by its semantics, rather than by its proof theory, as is the case with linear logic. We have trailed a thread from the categorical view of the proofs of BI as a bicartesian doubly closed category (DCC), through the proof theory of BI — both the sequent calculus LBI and our own contribution, the system SBISg in the calculus of structures, through to the Kripke resource semantics of BI by way of our proofs of the soundness and completeness of SBISg, and eventually to the hybrid standpoint. The work on SBISg showed how to extend Tiu's treatment of intuitionistic logic in the calculus of structures to BI, which is an intuitionistic logic extended with multiplicative conjunction and implication. In particular, we have demonstrated — essentially by way of the soundness and completeness proofs — that a two-valued system of polarity is sufficient for the proof theory of BI in the calculus of structures, despite the presence of two kinds of implication. (Four polarities, or a system of arbitrarily complex polarity labels, might have been reasonable guesses, and would have made the proof theory immensely complicated.) This work attests to the versatility and naturalness of the calculus of structures, and also to the naturalness of BI; indeed we think that it expresses the essential structure of BI more perspicuously than do other formalisms. Our soundness proof showed how to handle deep inference in a semantic setting, when dealing with the calculus of structures. Our work on the completeness proof permitted us to study a common approach to completeness for logics of intuitionistic character; and to look closely at the complications surrounding the handling of inconsistency in BI, and at the fine structure of the Kripke resource semantics.

Bibliography

Areces, C. [2004–]. The Hybrid Logics Home Page. `http://hylo.loria.fr`.

Areces, C. & Blackburn, P. [2001]. Bringing them all together, *Journal of Logic and Computation* **11**(5): 657–669.

Areces, C., Blackburn, P. & Marx, M. [2001]. Hybrid logics: Characterization, interpolation and complexity, *The Journal of Symbolic Logic* **66**(3): 977–1010.

Areces, C. & ten Cate, B. [2006]. Hybrid logics, *in* Blackburn, van Benthem & Wolter [2006].

Armelín, P. A. & Pym, D. J. [2001]. Bunched logic programming (Extended abstract), *in* R. Goré, A. Leitsch & T. Nipkow (eds), *Automated Reasoning: First International Joint Conference, IJCAR 2001. Siena, Italy, June 18–23, 2001, Proceedings*, Vol. 2083 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, pp. 289–304.

Béal, F., Méry, D. & Galmiche, D. [n.d.]. BILL: A theorem prover for propositional BI logic. Version 1.01b. `http://www.loria.fr/~dmery/BILL/`.

Bierman, G. M. & de Paiva, V. C. V. [2000]. On an intuitionistic modal logic, *Studia Logica* **65**: 383–416.

Biri, N. & Galmiche, D. [2003]. A separation logic for resource distribution. Extended abstract, *in* P. K. Pandya & J. Radhakrishnan (eds), *FST TCS 2003: Foundations of Software Technology and Theoretical Computer Science. Bombay, India, December 2003*, Vol. 2914 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, pp. 23–37.

Blackburn, P. [2000a]. Internalizing labelled deduction, *Journal of Logic and Computation* **10**(1): 137–168.

Blackburn, P. [2000b]. Representation, reasoning, and relational structures: a hybrid logic manifesto, *Logic Journal of the IGPL* **8**(3): 339–365.

Blackburn, P., de Rijke, M. & Venema, Y. [2001]. *Modal Logic*, Vol. 53 of *Cambridge Tracts in Theoretical Computer Science*, Cambridge University Press, Cambridge.

Blackburn, P. & Seligman, J. [1995]. Hybrid languages, *Journal of Logic, Language and Information* **4**(3): 251–272.

Blackburn, P. & Seligman, J. [1998]. What are hybrid languages?, *in* M. Kracht, M. de Rijke, H. Wansing & M. Zakharyaschev (eds), *Advances in Modal Logic, Volume 1*, number 87 in *CSLI Lecture Notes*, CSLI Publications, Stanford, pp. 41–62.

Blackburn, P. & van Benthem, J. [2006]. Modal logic: A semantic perspective, *in* Blackburn et al. [2006].

Blackburn, P., van Benthem, J. & Wolter, F. (eds) [2006]. *Handbook of Modal Logic*, Vol. 3 of *Studies in Logic and Practical Reasoning*, Elsevier.

Boolos, G. S. & Jeffrey, R. C. [1989]. *Computability and Logic*, third edn, Cambridge University Press, Cambridge.

Braüner, T. [2004a]. Natural deduction for hybrid logic, *Journal of Logic and Computation* **14**(3): 329–353.

Braüner, T. [2004b]. Two natural deduction systems for hybrid logic: A comparison, *Journal of Logic, Language and Information* **13**(1): 1–23.

Braüner, T. & de Paiva, V. [2003]. Towards constructive hybrid logic (Extended abstract), *Methods for Modalities 3, LORIA, Nancy, France, September 22–23, 2003. Proceedings*.

Braüner, T. & de Paiva, V. [2006]. Intuitionistic hybrid logic, *Journal of Applied Logic* **4**(3): 231–255.

Brünnler, K. [2004]. Intuitionistic logic in the calculus of structures. Unpublished note. `http://www.iam.unibe.ch/~kai/Current/minimal.html`.

Brünnler, K. [2006]. Locality for classical logic. To appear in the *Notre Dame Journal of Formal Logic*.

Chadha, R., Macedonio, D. & Sassone, V. [2006]. A hybrid intuitionistic logic: Semantics and decidability, *Journal of Logic and Computation* **16**(1): 27–59.

Davey, B. A. & Priestly, H. A. [2002]. *Introduction to Lattices and Order*, second edn, Cambridge University Press, Cambridge.

Demri, S. & Goré, R. [2002]. Display calculi for nominal tense logics, *Journal of Logic and Computation* **12**(6): 993–1016.

Došen, K. [1993]. A historical introduction to substructural logics, *in* P. Schroeder-Heister & K. Došen (eds), *Substructural Logics*, Vol. 2 of *Studies in Logic and Computation*, Clarendon Press, Oxford, pp. 1–30.

Dummett, M. [2000]. *Elements of Intuitionism*, Vol. 39 of *Oxford Logic Guides*, second edn, Clarendon Press, Oxford.

Ewald, W. B. [1986]. Intuitionistic tense and modal logic, *The Journal of Symbolic Logic* **51**(1): 166–179.

Feferman, S., Dawson, Jr., J. W., Kleene, S. C., Moore, G. H., Solovay, R. M. & van Heijenoort, J. (eds) [1986]. *Kurt Gödel: Collected Works, Volume I. Publications 1929–1936*, Oxford University Press, New York.

Fitting, M. [1973]. Model existence theorems for modal and intuitionistic logics, *The Journal of Symbolic Logic* **38**(4): 613–627.

Fitting, M. [1983]. *Proof Methods for Modal and Intuitionistic Logics*, Vol. 169 of *Synthese Library*, D. Reidel, Dordrecht.

Galmiche, D. & Méry, D. [2001]. Proof-search and countermodel generation in propositional BI logic. Extended abstract, *in* N. Kobayashi & B. C. Pierce (eds), *Theoretical Aspects of Computer Software: 4th International Symposium, TACS 2001. Sendai, Japan, October 29-31, 2001. Proceedings*, Vol. 2215 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, pp. 263–282.

Galmiche, D. & Méry, D. [2002]. Connection-based proof search in propositional BI logic, *in* A. Voronkov (ed.), *Automated Deduction – CADE-18: 18th International Conference on Automated Deduction. Copenhagen, Denmark, July 27-30, 2002. Proceedings*, Vol. 2392 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, pp. 111–128.

Galmiche, D. & Méry, D. [2003]. Semantic labelled tableaux for propositional $\mathbf{BI}^{\perp}$, *Journal of Logic and Computation* **13**(5): 707–753.

Galmiche, D. & Méry, D. [2004]. Resource graphs and countermodels in resource logics, *in* W. Ahrendt, P. Baumgartner & H. de Nivelle (eds), *Workshop on Disproving: Non-Theorems, Non-validity, Non-Provability. Second International Joint Conference on Automated Reasoning, IJCAR 2004. Cork, Ireland, July 5, 2004*, pp. 59–75. Proceedings published electronically `http://www.cs.chalmers.se/˜ahrendt/ijcar-ws-disproving/WS1.pdf`.

Galmiche, D. & Méry, D. [2005]. Resource graphs and countermodels in resource logics, *Electronic Notes in Theoretical Computer Science* **125**(3): 117–135.

Galmiche, D., Méry, D. & Pym, D. [2002]. Resource tableaux (Extended abstract), *in* J. Bradfield (ed.), *Computer Science Logic: 16th International Workshop, CSL 2002. 11th Annual Conference of the EACSL, Edinburgh. September 22–25, 2002. Proceedings*, Vol. 2471 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, pp. 183–199.

Galmiche, D., Méry, D. & Pym, D. [2005]. The semantics of BI and resource tableaux, *Mathematical Structures in Computer Science* **15**(6): 1033–1088.

Gentzen, G. [1934–35]. Untersuchungen über das logische Schließen, *Mathematische Zeitschrift* **39**: 176–210, 405–431. Translated as Investigations into logical deduction, in Szabo [1969, pp. 68–131].

Girard, J.-Y. [1987a]. Linear logic, *Theoretical Computer Science* **50**(1): 1–101.

Girard, J.-Y. [1987b]. *Proof Theory and Logical Complexity, Volume 1*, Vol. 1 of *Studies in Proof Theory, Monographs*, Bibliopolis, Naples.

Girard, J.-Y. [1993]. On the unity of logic, *Annals of Pure and Applied Logic* **59**: 201–217.

Girard, J.-Y. [1995]. Linear logic: its syntax and semantics, *in* J.-Y. Girard, Y. Lafont & L. Regnier (eds), *Advances in Linear Logic*, Vol. 222 of *London Mathematical Society Lecture Notes Series*, Cambridge University Press, Cambridge, pp. 1–42.

Girard, J.-Y., Taylor, P. & Lafont, Y. [1989]. *Proofs and Types*, Vol. 7 of *Cambridge Tracts in Theoretical Computer Science*, Cambridge University Press, Cambridge.

Gödel, K. [1933]. Eine Interpretation des intuitionistischen Aussagenkalküls, *Ergebnisse eines mathematischen Kolloquiums* **4**: 39–40. Reprinted with introductory note by A. S. Troelstra, and translated as An interpretation of the intuitionistic propositional calculus, in Feferman, Dawson, Jr., Kleene, Moore, Solovay & van Heijenoort [1986, pp. 296–303].

Guglielmi, A. [2004]. A system of interaction and structure, *Technical Report WV-02-10*, Technische Universität Dresden. To appear in *ACM Transactions on Computational Logic*.

Harland, J. A. & Pym, D. J. [2003]. Resource-distribution via Boolean constraints, *ACM Transactions on Computational Logic* **4**(1): 56–90.

Horsfall, B. [2006]. Towards BI in the calculus of structures. Unpublished paper.

Hughes, G. E. & Cresswell, M. J. [1996]. *A New Introduction to Modal Logic*, Routledge, London.

Ishtiaq, S. & O'Hearn, P. W. [2001]. BI as an assertion language for mutable data structures, *Proceedings of the 28th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages. London, 2001*, ACM Press, New York, pp. 14–26.

Jia, L. & Walker, D. [2003]. Modal proofs as distributed programs, *Technical Report TR-671-03*, Princeton University.

Jia, L. & Walker, D. [2004]. Modal proofs as distributed programs (Extended abstract), *in* D. A. Schmidt (ed.), *Programming Languages and Systems: 13th European Symposium on Programming, ESOP 2004, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2004, Barcelona, Spain, March 29–April 2, 2004. Proceedings*, Vol. 2986 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, pp. 219–233.

Kripke, S. A. [1959]. Semantical analysis of modal logic (Abstract), *The Journal of Symbolic Logic* **24**(4): 323–324.

Kripke, S. A. [1963a]. Semantical analysis of modal logic I: Normal modal propositional calculi, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **9**: 67–96.

Kripke, S. A. [1963b]. Semantical considerations on modal logic, *Acta Philosophica Fennica* **16**: 83–94. Reprinted in Linsky [1971, pp. 63–72].

Kripke, S. A. [1965]. Semantical analysis of intuitionistic logic I, *in* J. N. Crossley & M. A. E. Dummett (eds), *Formal Systems and Recursive Functions: Proceedings of the Eighth Logic Colloquium. Oxford, July 1963*, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, pp. 92–130.

Lambek, J. [1958]. The mathematics of sentence structure, *The American Mathematical Monthly* **65**(3): 154–170.

Lambek, J. & Scott, P. J. [1986]. *Introduction to higher order categorical logic*, Vol. 7 of *Cambridge studies in advanced mathematics*, Cambridge University Press, Cambridge.

Lincoln, P. [1992]. Linear logic, *ACM SIGACT News* **23**(2): 29–37.

Lincoln, P., Mitchell, J., Scedrov, A. & Shankar, N. [1992]. Decision problems for propositional linear logic, *Annals of Pure and Applied Logic* **56**(1–3): 239–311.

Linsky, L. (ed.) [1971]. *Reference and Modality*, Oxford Readings in Philosophy, Oxford University Press, Oxford.

Mac Lane, S. [1998]. *Categories for the Working Mathematician*, Vol. 5 of *Graduate Texts in Mathematics*, second edn, Springer-Verlag, New York.

Negri, S. & von Plato, J. [2001]. *Structural Proof Theory*, Cambridge University Press, Cambridge.

O'Hearn, P. W. [1999]. Resource interpretations, bunched implications and the $\alpha\lambda$-calculus (Preliminary version), *in* J.-Y. Girard (ed.), *Typed Lambda Calculi and Applications, TLCA'99*, Vol. 1581 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, pp. 258–279.

O'Hearn, P. W. [2003]. On bunched typing, *Journal of Functional Programming* **13**(4): 747–796.

O'Hearn, P. W. [2005]. Resources, concurrency and local reasoning. To appear in *Theoretical Computer Science*.

O'Hearn, P. W. & Pym, D. J. [1999]. The logic of bunched implications, *The Bulletin of Symbolic Logic* **5**(2): 215–244.

O'Hearn, P. W., Reynolds, J. C. & Yang, H. [2001]. Local reasoning about programs that alter data structures, *in* L. Fribourg (ed.), *Computer Science Logic: 15th International Workshop, CSL 2001. 10th Annual Conference of the EACSL. Paris, France, September 10–13, 2001. Proceedings*, Vol. 2142 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, pp. 1–19.

Ono, H. & Komori, Y. [1985]. Logics without the contraction rule, *The Journal of Symbolic Logic* **50**(1): 169–201.

Passy, S. & Tinchev, T. [1991]. An essay in combinatory dynamic logic, *Information and Computation* **93**(2): 263–332.

Pinto, L. & Dyckhoff, R. [1995]. Loop-free construction of counter-models for intuitionistic propositional logic, *in* M. Behara, R. Fritsch & R. G. Lintz (eds), *Proceedings of the 2nd Gauss Symposium. Conference A: mathematics and theoretical physics. Munich, Germany, August 2–7, 1993*, Walter de Gruyter, Berlin, pp. 225–232.

Prawitz, D. [1965]. *Natural Deduction: A Proof-Theoretical Study*, Vol. 3 of *Stockholm Studies in Philosophy*, Almqvist & Wiksell, Stockholm.

Prawitz, D. [1971]. Ideas and results in proof theory, *in* J. E. Fenstad (ed.), *Proceedings of the Second Scandinavian Logic Symposium*, Vol. 63 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, pp. 235–307.

Prawitz, D. [2006]. Meaning approached via proofs, *Synthese* **148**(3): 507–524.

Prior, A. [1967]. *Past, Present and Future*, Clarendon Press, Oxford.

Pym, D. J. [1999]. On bunched predicate logic, *14th Symposium on Logic in Computer Science, LICS '99*, IEEE Computer Society Press, pp. 183–192.

Pym, D. J. [2002]. *The Semantics and Proof Theory of the Logic of Bunched Implications*, Vol. 26 of *Applied Logic Series*, Kluwer Academic Publishers, Dordrecht.

Pym, D. J. [2006]. Errata and remarks for *The Semantics and Proof Theory of the Logic of Bunched Implications*. `http://www.cs.bath.ac.uk/~pym/BI-monograph-errata.pdf`.

Pym, D. J., O'Hearn, P. W. & Yang, H. [2004]. Possible worlds and resources: the semantics of **BI**, *Theoretical Computer Science* **315**(1): 257–305.

Restall, G. [2000]. *An Introduction to Substructural Logics*, Routledge, London.

Reynolds, J. C. [2002]. Separation logic: A logic for shared mutable data structures, *Proceedings of the 17th Annual IEEE Symposium on Logic in Computer Science. July 22–25, 2002. Copenhagen, Denmark*, IEEE Computer Society Press, pp. 55–74.

Routley, R. & Meyer, R. K. [1972]. The semantics of entailment – II, *Journal of Philosophical Logic* **1**: 53–73.

Seligman, J. [1991]. A cut-free sequent calculus for elementary situated reasoning, *Research Paper HCRC-RP 22*, Human Communication Research Centre, University of Edinburgh.

Seligman, J. [1997]. The logic of correct description, *in* M. de Rijke (ed.), *Advances in Intensional Logic*, Vol. 7 of *Applied Logic Series*, Kluwer Academic Publishers, Dordrecht, pp. 107–135.

Seligman, J. [2001]. Internalization: The case of hybrid logics, *Journal of Logic and Computation* **11**(5): 671–689.

Simpson, A. K. [1994]. *The Proof Theory and Semantics of Intuitionistic Modal Logic*, PhD thesis, University of Edinburgh.

Straßburger, L. [2003]. MELL in the calculus of structures, *Theoretical Computer Science* **309**(1–3): 213–285.

Szabo, M. E. (ed.) [1969]. *The Collected Papers of Gerhard Gentzen*, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam.

Tiu, A. [2005]. A local system for intuitionistic logic: Preliminary results. Unpublished draft.

Tiu, A. [2006]. A local system for intuitionistic logic. Accepted for the *13th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR 2006)*, November 13–17, 2006, Phnom Penh, Cambodia. Full paper, with proofs `http://users.rsise.anu.edu.au/~tiu/papers/localint.pdf`.

Troelstra, A. S. & Schwichtenberg, H. [2000]. *Basic Proof Theory*, Vol. 43 of *Cambridge Tracts in Theoretical Computer Science*, second edn, Cambridge University Press, Cambridge.

Urquhart, A. [1972]. Semantics for relevant logics, *The Journal of Symbolic Logic* **37**(1): 159–169.

van Dalen, D. [2004]. *Logic and Structure*, Universitext, fourth edn, Springer-Verlag, Berlin.

Walker, D. [2004]. Substructural type systems, *in* B. C. Pierce (ed.), *Advanced Topics in Types and Programming Languages*, The MIT Press, Cambridge, Massachusetts, pp. 3–43.

Wolter, F. & Zakharyaschev, M. [1999]. Intuitionistic modal logic, *in* A. Cantini, E. Casari & P. Minari (eds), *Logic and Foundations of Mathematics*, Vol. 280 of *Synthese Library*, Kluwer Academic Publishers, Dordrecht, pp. 227–238.

Appendix A

Cut-elimination for propositional linear logic

Cut-elimination is an established result for linear logic, proven for proof-nets by Girard [1987a] and for a sequent calculus by Lincoln, Mitchell, Scedrov & Shankar [1992]. We set out to construct from first principles a detailed proof for the right-sided sequent calculus presented by Girard [1995] (see Figure A.1), using established techniques for sequent calculi generally [Negri & von Plato 2001]. For cases involving *contraction* we follow Lincoln et al. [1992] in the use of the derived rule **Cut!**, or *multicut*.

The *multicut* rule is admissible in the sequent calculus with *cut*. It is used to simplify the proof (and it turns out to be indispensable in implementing a cut-elimination algorithm, to avoid difficulties in representing indefinite sequences of *contraction*s). We use it in the following way. We write $(?A)^n$ as an abbreviation for n consecutive occurrences of $?A$. When we encounter a derivation with several contractions preceding a cut:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, (?\phi)^n}}{\Rightarrow \Gamma, (?\phi)^{n-1}}\,\textit{contraction}}{\vdots}\,\textit{contraction}}{\Rightarrow \Gamma, ?\phi} \qquad \dfrac{\vdots}{\Rightarrow !\phi^{\perp}, \Delta}}{\Rightarrow \Gamma, \Delta}\,\textit{cut}$$

we rewrite it as an occurrence of the *multicut* rule,[1] thus:

[1] A *degenerate multicut*, with $n = 1$, is simply a *cut*.

$$\frac{}{\Rightarrow \phi, \phi^\perp}\ \textit{identity} \qquad \frac{\Rightarrow \Gamma, \phi \qquad \Rightarrow \phi^\perp, \Delta}{\Rightarrow \Gamma, \Delta}\ \textit{cut} \qquad \frac{\Rightarrow \Gamma}{\Rightarrow \Gamma'}\ \textit{exchange}$$

$$\frac{}{\Rightarrow \mathbf{1}}\ \textit{one} \qquad \textit{No rule introduces } \mathbf{0} \qquad \frac{}{\Rightarrow \Gamma, \top}\ \textit{true} \qquad \frac{\Rightarrow \Gamma}{\Rightarrow \Gamma, \bot}\ \textit{false}$$

$$\frac{\Rightarrow \Gamma, \phi \qquad \Rightarrow \psi, \Delta}{\Rightarrow \Gamma, \phi \otimes \psi, \Delta}\ \textit{times} \qquad \frac{\Rightarrow \Gamma, \phi, \psi}{\Rightarrow \Gamma, \phi \parr \psi}\ \textit{par}$$

$$\frac{\Rightarrow \Gamma, \phi \qquad \Rightarrow \Gamma, \psi}{\Rightarrow \Gamma, \phi \,\&\, \psi}\ \textit{with} \qquad \frac{\Rightarrow \Gamma, \phi}{\Rightarrow \Gamma, \phi \oplus \psi}\ \textit{left plus} \qquad \frac{\Rightarrow \Gamma, \psi}{\Rightarrow \Gamma, \phi \oplus \psi}\ \textit{right plus}$$

$$\frac{\Rightarrow ?\Gamma, \phi}{\Rightarrow ?\Gamma, !\phi}\ \textit{promotion} \qquad \frac{\Rightarrow \Gamma}{\Rightarrow \Gamma, ?\phi}\ \textit{weakening}$$

$$\frac{\Rightarrow \Gamma, \phi}{\Rightarrow \Gamma, ?\phi}\ \textit{dereliction} \qquad \frac{\Rightarrow \Gamma, ?\phi, ?\phi}{\Rightarrow \Gamma, ?\phi}\ \textit{contraction}$$

Figure A.1: The right-sided sequent calculus for linear logic

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\mathbf{1}^\perp & := & \bot & \bot^\perp & := & \mathbf{1} \\
\top^\perp & := & \mathbf{0} & \mathbf{0}^\perp & := & \top \\
(p)^\perp & := & p^\perp & (p^\perp)^\perp & := & p \\
(\phi \otimes \psi)^\perp & := & \phi^\perp \parr \psi^\perp & (\phi \parr \psi)^\perp & := & \phi^\perp \otimes \psi^\perp \\
(\phi \,\&\, \psi)^\perp & := & \phi^\perp \oplus \psi^\perp & (\phi \oplus \psi)^\perp & := & \phi^\perp \,\&\, \psi^\perp \\
(!\phi)^\perp & := & ?\phi^\perp & (?\phi)^\perp & := & !\phi^\perp
\end{array}$$

$$\phi \multimap \psi := \phi^\perp \parr \psi$$

Figure A.2: Defined connectives linear negation and implication

$$\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, (?\phi)^n} \qquad \dfrac{\vdots}{\Rightarrow\, !\phi^\perp, \Delta}}{\Rightarrow \Gamma, \Delta}\ \textit{multicut}\ (n \geq 1)$$

In fact, we can say that *multicut* is simply a compact representation of derivations of this form. After all, the cut-elimination procedure is applied to proofs that are already known to be valid, so every rewritten derivation is known to be valid, and the *multicut* representation has the same premises and the same conclusion. In this sense, we do not need to define admissibility of *multicut*, but merely regard it as syntactically defined. But if we regard it a rule new rule which we would like to add *conservatively* to the system, we do need to prove admissibility. Thinking in this way, we consider how a *multicut* may be replaced, rather than rewritten.

Theorem 5 *The multicut rule in admissible in the sequent calculus, that is, any proof containing a multicut can be transformed into a proof not containing a multicut.* □

PROOF The proof is by induction on n. An occurrence of *multicut* in a derivation can always be replaced as follows, with the same premises and conclusion, and preserving validity:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, (?\phi)^n}}{\Rightarrow \Gamma, (?\phi)^{n-1}}\ \textit{contraction} \qquad \dfrac{\vdots}{\Rightarrow\, !\phi^\perp, \Delta}}{\Rightarrow \Gamma, \Delta}\ \textit{cut}$$

until $n = 1$, in which case the replacement *multicut* is simply a *cut*. ■

Variants of *multicut* are used in cut-elimination proofs other than for linear logic. Exponential operators are peculiar to linear logic, but *multicut* can be used to handle multiple occurrences of *contraction*.

Definition 37 Weight is an inductively-defined measure of the complexity of a formula:

$$\begin{aligned}
&\mathit{weight}(p) = \mathit{weight}(\bot) = \mathit{weight}(\top) = \mathit{weight}(\mathbf{0}) = \mathit{weight}(\mathbf{1}) = 1\\
&\mathit{weight}(\phi^\perp) = \mathit{weight}(\phi)\\
&\mathit{weight}(\phi \circ \psi) = \mathit{weight}(\phi) + \mathit{weight}(\psi) + 1
\end{aligned}$$

where p is any propositional variable or constant, and $\circ$ is any binary connective. Cut-weight is the weight of the cut-formula in an application of the *cut* rule. The height of a derivation (or rule application) is the sum of the heights of the derivations of each of the premises of the derivation, plus one. Cut-height is the height of an application of the *cut* rule. The operator $\succ$ between a pair of proofs means "reduces to", but says nothing about cut elimination or reduction in cut-weight or -height; $\succ_e$ means "reduces, eliminating the cut, to"; $\succ_w$ means "reduces, with reduced cut-weight for all replacement cuts, to"; and $\succ_h$ means "reduces, with reduced cut-height for all replacement cuts, to". $\square$

Theorem 6 (Cut Elimination) *If a sequent of propositional linear logic can be proven using the* cut *rule, it can be proven without using the* cut *rule.* $\square$

PROOF We give a procedure by which any proof containing applications of the *cut* rule can be transformed into a cut-free proof. The cut-elimination procedure is defined inductively. At each step, a cut in a proof is replaced either with a cut-free derivation, or by one or more applications of cut, having strictly smaller cut-weight or cut-height. Cut-weight and cut-height are obviously well-founded measures. Cases are divided into several groups: (i) either premise of the cut is an axiom; (ii) both premises are principal in their derivations; (iii) one premise is principal in its derivation; (iv) neither premise is principal in its derivation. Cases in group (i) remove a cut outright; in group (ii) replace the cut with one or more cuts of strictly smaller weight; in (iii) and (iv) replace the cut with one or more cuts of strictly smaller height; in these cases we are 'permuting the cut upwards'. (In a number of cases, there is more than one way, modulo *exchange*, to replace a cut, and hence the procedure lacks the Church-Rosser property.)

(i) EITHER PREMISE IS AN AXIOM. The following are all the cases where at least one premise is an axiom[2] *modulo* switching left and right cut premises, rewriting negated formulæ,[3] and interposing exchanges:

[2] "Axiom" is an abuse of terminology; we mean a rule having no premises.

[3] Negation $(\,)^\perp$ in linear logic is recursively defined by rewriting rules. For example, $(\phi \otimes \psi)^\perp := \phi^\perp \parr \psi^\perp$.

$$\dfrac{\dfrac{}{\Rightarrow \phi, \phi^{\perp}}\,\mathit{identity} \qquad \dfrac{\vdots}{\Rightarrow \phi, \Delta}}{\Rightarrow \phi, \Delta}\,\mathit{cut} \quad \succ_e \quad \dfrac{\vdots}{\Rightarrow \phi, \Delta}$$

When one premise is the conclusion of *true*, the other premise must be the conclusion of *identity*: there is no other way to introduce **0**. Hence this is just a special case of the *identity* elimination.

$$\dfrac{\dfrac{}{\Rightarrow \Gamma, \top}\,\mathit{true} \qquad \dfrac{}{\Rightarrow \mathbf{0}, \top}\,\mathit{identity}}{\Rightarrow \Gamma, \top}\,\mathit{cut} \quad \succ_e \quad \dfrac{}{\Rightarrow \Gamma, \top}\,\mathit{true}$$

There are no cases for one, since this rule concludes in a sequent with only one formula in it. The rule for *false* is a special case, but not actually an axiomatic case. The negated cut formula on the right must be **1**, which can only be introduced in a singleton sequent, hence the cut delta must be empty or contain only bottoms, introduced by false and why not-formulæ, introduced by weakening. If the cut delta is non-empty, the **1** is non-principal, hence the cut will be permuted upwards, until a singleton **1** is reached.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma}}{\Rightarrow \Gamma, \perp}\,\mathit{false} \qquad \dfrac{}{\Rightarrow \mathbf{1}}\,\mathit{one}}{\Rightarrow \Gamma}\,\mathit{cut} \quad \succ_e \quad \dfrac{\vdots}{\Rightarrow \Gamma}$$

There is another possibility, where why not-formulæ are introduced following the introduction of **1** by applications of *weakening*. In these cases, *weakening* can always be applied to reach the eventual conclusion $\Rightarrow \Gamma, ?\Delta$, given $\Rightarrow \Gamma$. But strictly speaking, the cut-formula is not principal on the right in this case, which would in any case dealt with less directly.

(ii) THE CUT-FORMULA IS PRINCIPAL FOR BOTH PREMISES. A double line indicates that an application of exchange may be required. In both possible reductions in this case, the cut-weight for each of the replacement cuts it strictly less than the original cut-weight.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Gamma, \phi, \psi$}
\RightLabel{\textit{par}}
\UnaryInfC{$\Rightarrow \Gamma, \phi \parr \psi$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Delta, \phi^{\perp}$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \psi^{\perp}, Z$}
\doubleLine
\RightLabel{\textit{times}}
\BinaryInfC{$\Rightarrow \phi^{\perp} \otimes \psi^{\perp}, \Delta, Z$}
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, \Delta, Z$}
\end{prooftree}
\quad \succ_w
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Gamma, \phi, \psi$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \psi^{\perp}, Z$}
\doubleLine
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, Z, \phi$}
\AxiomC{$\vdots$}
\doubleLine
\UnaryInfC{$\Rightarrow \phi^{\perp}, \Delta$}
\doubleLine
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, \Delta, Z$}
\end{prooftree}
$$

$$
\textit{OR} \quad
\begin{prooftree}
\AxiomC{$\vdots$}
\doubleLine
\UnaryInfC{$\Rightarrow \Gamma, \psi, \phi$}
\AxiomC{$\vdots$}
\doubleLine
\UnaryInfC{$\Rightarrow \phi^{\perp}, \Delta$}
\doubleLine
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, \Delta, \psi$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \psi^{\perp}, Z$}
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, \Delta, Z$}
\end{prooftree}
$$

The case with a *times* cut-formula on the left is the mirror-image of the preceding case, and is omitted.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Gamma, \phi$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Gamma, \psi$}
\RightLabel{\textit{with}}
\BinaryInfC{$\Rightarrow \Gamma, \phi \mathbin{\&} \psi$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Delta, \phi^{\perp}$}
\doubleLine
\RightLabel{\textit{left plus}}
\UnaryInfC{$\Rightarrow \phi^{\perp} \oplus \psi^{\perp}, \Delta$}
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, \Delta$}
\end{prooftree}
\quad \succ_w \quad
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Gamma, \phi$}
\AxiomC{$\vdots$}
\doubleLine
\UnaryInfC{$\Rightarrow \phi^{\perp}, \Delta$}
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, \Delta$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Gamma, \phi$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Gamma, \psi$}
\RightLabel{\textit{with}}
\BinaryInfC{$\Rightarrow \Gamma, \phi \mathbin{\&} \psi$}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Delta, \psi^{\perp}$}
\doubleLine
\RightLabel{\textit{right plus}}
\UnaryInfC{$\Rightarrow \phi^{\perp} \oplus \psi^{\perp}, \Delta$}
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, \Delta$}
\end{prooftree}
$$

$$
\succ_w \quad
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$\Rightarrow \Gamma, \psi$}
\AxiomC{$\vdots$}
\doubleLine
\UnaryInfC{$\Rightarrow \psi^{\perp}, \Delta$}
\RightLabel{\textit{cut}}
\BinaryInfC{$\Rightarrow \Gamma, \Delta$}
\end{prooftree}
$$

The cases with a *plus* cut-formula on the left are the mirror-images of the preceding case, and are omitted.

Now we treat the exponential rules.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \phi}}{\Rightarrow \Gamma, ?\phi}\,\textit{dereliction} \qquad \dfrac{\dfrac{\vdots}{\Rightarrow ?\Delta, \phi^{\perp}}}{\Rightarrow !\phi^{\perp}, ?\Delta}\,\textit{promotion}}{\Rightarrow \Gamma, ?\Delta}\,\textit{cut} \quad \succ_w \quad \dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \phi} \qquad \dfrac{\vdots}{\Rightarrow \phi^{\perp}, ?\Delta}}{\Rightarrow \Gamma, ?\Delta}\,\textit{cut}$$

But when we encounter a *multicut*:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, (?\phi)^{n-1}, \psi}}{\Rightarrow \Gamma, (?\phi)^{n}}\,\textit{dereliction} \qquad \dfrac{\dfrac{\vdots}{\Rightarrow ?\Delta, \phi^{\perp}}}{\Rightarrow !\phi^{\perp}, ?\Delta}\,\textit{promotion}}{\Rightarrow \Gamma, ?\Delta}\,\textit{multicut} \quad \succ$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \phi, (?\phi)^{n-1}} \qquad \dfrac{\dfrac{\vdots}{\Rightarrow ?\Delta, \phi^{\perp}}}{\Rightarrow !\phi^{\perp}, ?\Delta}\,\textit{promotion}}{\Rightarrow \Gamma, ?\Delta, \phi}\,\textit{multicut} \qquad \dfrac{\vdots}{\Rightarrow \phi^{\perp}, ?\Delta}}{\Rightarrow \Gamma, ?\Delta, ?\Delta}\,\textit{cut}}{\vdots}\,\textit{weakening}}{\Rightarrow \Gamma, ?\Delta}$$

The lower replacement *cut* is of strictly lesser weight than the original multicut, but the replacement *multicut* does not always decrease the cut height, that is, if the derivation of the right premise was equal to or greater in height than the derivation of the left premise. Instead, we have to perform an induction on *n*, which will eventually reach 1 after repeated applications of this reduction, that is, the simpler *cut* case will be reached, and cut-weight will be strictly reduced.

(iii) Only the left premise has cut-formula as principal. This encompasses cases where the right-hand premise is the conclusion of a *cut*. In these cases we permute the cut upwards. We divide the cases in two, depending on whether the right-premise derivation has one or two premises. There is no further need to treat individual connectives separately.

$$\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \chi}\,L \qquad \dfrac{\dfrac{\vdots}{\Rightarrow \chi^{\perp}, \Delta}}{\Rightarrow \chi^{\perp}, \Delta'}\,R_1}{\Rightarrow \Gamma, \Delta'}\,cut \quad \succ \quad \dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \chi}\,L \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp}, \Delta}}{\Rightarrow \Gamma, \Delta}\,cut}{\Rightarrow \Gamma, \Delta'}\,R_1$$

$$\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \chi}\,L \qquad \dfrac{\dfrac{\vdots}{\Rightarrow \chi^{\perp}, \Delta} \quad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \chi^{\perp}, \Delta', Z'}\,R_2}{\Rightarrow \Gamma, \Delta', Z'}\,cut \quad \succ \quad \dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \chi}\,L \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp}, \Delta}}{\Rightarrow \Gamma, \Delta}\,cut \qquad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \Gamma, \Delta', Z'}\,R_2$$

There is a trivial variant of the second case, where the cut-formula originates in the right premise of R_2. If R_2 is *cut*, its new application does not decrease in height or weight, but it is permuted downwards, and will be dealt with directly in a subsequent visit. But there is a difficultly. The cut-height is not actually reduced where the height of the derivation of the left premise is greater than or equal to the height of the derivation of the right premise. The height of the derivation of the right premise does strictly decrease, however, but it is necessary to invoke another induction. When the replacement *cut* is visited again, it will be either:

1. Under one of the present cases, in which case the height of the derivation of the right premise of the replacement *cut* will again decrease; repetition of this case must eventually lead to one of the remaining cases, because height strictly decreases. This includes the case where the right premise of the original *cut* is the conclusion of a *cut*;

2. (a) $C^{\perp}$ is principal in the derivation of the right premise of the replacement cut: in this case, the cut-formula will be principal in the derivation of both premises, and we have already established that for all of these cases, cut-weight strictly decreases;

   (b) The right premise is an axiom, and the cut is eliminated.

(iv) ONLY THE RIGHT PREMISE HAS CUT-FORMULA AS PRINCIPAL.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \chi}}{\Rightarrow \Gamma', \chi}\,L_1 \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp}, \Delta}}{\Rightarrow \Gamma', \Delta}\,cut \quad \succ \quad \dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma, \chi} \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp}, \Delta}\,R}{\Rightarrow \Gamma, \Delta}\,cut}{\Rightarrow \Gamma', \Delta}\,L_1$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \qquad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \Gamma',\chi}\,L_2 \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}\,R}{\Rightarrow \Gamma',\Delta}\,cut \quad \succ \quad \dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}\,R}{\Rightarrow \Delta,\Gamma}\,cut \qquad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \Gamma',\Delta}\,L_2$$

The argument proceeds as above.

(v) Neither premise has cut-formula as principal. In all of the following cases, cut-height is strictly reduced. If left and right premises are both conclusions of one-premise rules, there are two possible outcomes; either order of left and right rules is acceptable.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi}}{\Rightarrow \Gamma',\chi}\,L_1 \qquad \dfrac{\dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \chi^{\perp},\Delta'}\,R_1}{\Rightarrow \Gamma',\Delta'}\,cut \quad \succ$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \Gamma,\Delta}\,cut}{\Rightarrow \Gamma',\Delta}\,L_1}{\Rightarrow \Gamma',\Delta'}\,R_1 \quad OR \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \Gamma,\Delta}\,cut}{\Rightarrow \Gamma,\Delta'}\,R_1}{\Rightarrow \Gamma',\Delta'}\,L_1$$

If left and right premises are conclusions of rules with different numbers of premises, there are also two possible outcomes, *modulo* trivial variations.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi}}{\Rightarrow \Gamma',\chi}\,L_1 \qquad \dfrac{\dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta} \qquad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \chi^{\perp},\Delta'}\,R_2}{\Rightarrow \Gamma',\Delta'}\,cut \quad \succ$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \Gamma,\Delta}\,cut}{\Rightarrow \Gamma',\Delta}\,L_1 \qquad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \Gamma',\Delta'}\,R_2 \quad OR \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \qquad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \Gamma,\Delta}\,cut \qquad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \Gamma,\Delta'}\,R_2}{\Rightarrow \Gamma',\Delta'}\,L_1$$

The L_2 / R_1 cases are a mirror-image of L_1 / R_2:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow Z} \quad \dfrac{\vdots}{\Rightarrow \Gamma,\chi}}{\Rightarrow \Gamma',\chi}\,L_2 \qquad \dfrac{\dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \chi^{\perp},\Delta'}\,R_1}{\Rightarrow \Gamma',\Delta'}\,\mathit{cut} \qquad \succ$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow Z} \quad \dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \quad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \Gamma,\Delta}\,\mathit{cut}}{\Rightarrow \Gamma',\Delta}\,L_2}{\Rightarrow \Gamma',\Delta'}\,R_1 \quad \mathit{OR} \quad \dfrac{\dfrac{\vdots}{\Rightarrow Z} \quad \dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \quad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \Gamma,\Delta}\,\mathit{cut}}{\Rightarrow \Gamma,\Delta'}\,R_1}{\Rightarrow \Gamma',\Delta'}\,L_2$$

If left and right premises are conclusions of rules both having two two premises, there are also two possible outcomes, *modulo* trivial variations.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \chi,\Gamma} \quad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \Gamma',\chi}\,L_2 \qquad \dfrac{\dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta} \quad \dfrac{\vdots}{\Rightarrow H}}{\Rightarrow \chi^{\perp},\Delta'}\,R_2}{\Rightarrow \Gamma',\Delta'}\,\mathit{cut} \qquad \succ$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \quad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \Gamma,\Delta}\,\mathit{cut} \quad \dfrac{\vdots}{\Rightarrow H}}{\Rightarrow \Delta',\Gamma}\,R_2 \quad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \Gamma',\Delta'}\,L_2$$

$$\mathit{OR} \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \Gamma,\chi} \quad \dfrac{\vdots}{\Rightarrow \chi^{\perp},\Delta}}{\Rightarrow \Delta,\Gamma}\,\mathit{cut} \quad \dfrac{\vdots}{\Rightarrow Z}}{\Rightarrow \Gamma',\Delta}\,L_2 \quad \dfrac{\vdots}{\Rightarrow H}}{\Rightarrow \Gamma',\Delta'}\,R_2$$

The alternative possibilities in these cases correspond to the order of application of the left and right rules from the original cut. There are further, less interesting, variations according to the origins of the cut-formula and its negation, which we omit. ■