Orbispaces, Configurations and Quasi-fibrations

Jeffrey Steven Bailes

July 2015

School of Mathematics and Statistics
The University of Melbourne

Submitted in total fulfilment of the requirements of the degree of Doctor of Philosophy
Abstract

The heart of this thesis tries to extend previous ideas about homological stability of configuration spaces on manifolds to the setting of orbifolds. When using the topological groupoid definition for an orbifold, there is a natural way to define the analogue of a configuration space.

Given an orbifold with boundary, the document works through defining a map which adds points to a configuration on its interior. This map is proved to induce an injective map on integral homology. With this result in hand, homological stability becomes the goal. While such a result does not appear in this work, some intermediary results do appear. Using a quasi-fibration criterion that is presented within, the hope is that this will form the foundations of future work in finding the stable homology for these objects.

Also appearing here is some investigative work on the Salvetti Complex. Looking at the specific case for the pure braid group, the document presents a concrete way to represent the Salvetti complex simplicially. The techniques here are then used in a reference Betti number calculation implementation, coded in Python.
Declaration

This is to certify that:

i. the thesis comprises only my original work towards the PhD except where indicated in the Preface;

ii. due acknowledgement has been made in the text to all other material used; and

iii. the thesis is fewer than 100 000 words in length, exclusive of tables, maps, bibliographies and appendices.
Acknowledgments

I’d like to thank my supervisor, Craig Westerland, for all of his guidance and help over the last three-and-a-half years. You helped immensely in keeping me calm and sane during the fifteen months I spent in Minnesota, discussing everything from beer to politics. Without your continual efforts, this work would not have come to fruition. I’d also like to thank Arun Ram for agreeing to be my co-supervisor when Craig moved overseas, and Alex Ghitza for joining with these two to form my PhD supervisory committee.

As for my fellow students, a big cheers to everyone I’ve had the pleasure to share an office with over the years, in Melbourne as well as Minnesota. The friendships I made during my time in the Melbourne University Mathematics and Statistics society assured I did not become too disconnected from reality.

Finally I’d like to thank my parents and family for their support during my degree. Mum, the look of puzzlement on your face every month after you asked me again for the title of my thesis was worth it in itself.
# Contents

<table>
<thead>
<tr>
<th>List of Figures</th>
<th>ix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2 Orbifold Background</td>
<td>7</td>
</tr>
<tr>
<td>2.1 Definitions and Preliminary Examples</td>
<td>7</td>
</tr>
<tr>
<td>2.1.1 Classical Orbifolds</td>
<td>7</td>
</tr>
<tr>
<td>2.1.2 The Groupoid Approach to Orbifolds</td>
<td>11</td>
</tr>
<tr>
<td>2.2 An Analogue of Mapping Spaces for Orbispaces</td>
<td>17</td>
</tr>
<tr>
<td>2.2.1 Orb-spaces</td>
<td>19</td>
</tr>
<tr>
<td>2.3 The Configuration Orbispace</td>
<td>29</td>
</tr>
<tr>
<td>2.4 Examples Relating to the Functor $R[G]$</td>
<td>39</td>
</tr>
<tr>
<td>3 Homological Injectivity</td>
<td>49</td>
</tr>
<tr>
<td>3.1 The Stabilisation Map</td>
<td>49</td>
</tr>
<tr>
<td>3.2 The Transfer Map</td>
<td>52</td>
</tr>
<tr>
<td>3.3 A Covering Map Up To Homotopy</td>
<td>54</td>
</tr>
<tr>
<td>3.4 Proof of Homological Injectivity</td>
<td>63</td>
</tr>
<tr>
<td>3.5 Main Result on Homological Injectivity</td>
<td>77</td>
</tr>
<tr>
<td>4 A Quasi-Fibration Criterion</td>
<td>79</td>
</tr>
<tr>
<td>4.1 Background</td>
<td>80</td>
</tr>
<tr>
<td>4.2 Homotopy Type of the Fibres</td>
<td>85</td>
</tr>
<tr>
<td>4.3 The Quasi-Fibration</td>
<td>92</td>
</tr>
<tr>
<td>4.4 Example: Configurations on Orbifolds</td>
<td>98</td>
</tr>
<tr>
<td>5 Relation to Mapping Spaces of Orbifolds</td>
<td>105</td>
</tr>
<tr>
<td>5.1 Already Known Results for Manifolds</td>
<td>105</td>
</tr>
<tr>
<td>5.2 Analogous Ideas on Orbifolds</td>
<td>108</td>
</tr>
</tbody>
</table>
List of Figures

1.1 Configurations on a cricket field ...................................... 1

2.1 The classical orbifold $D^2/\mathbb{Z}$ with reflection action .................. 9
2.2 The classical orbifold $D^2/\mathbb{Z}$ with rotation action ......................... 10
2.3 The teardrop orbifold .................................................. 10
2.4 Ghost points on $[D^2/\mathbb{Z}]$ ........................................... 31

3.1 Collar around the boundary of an orbifold .................................... 51
3.2 Pushing points in the collar away from the boundary ......................... 51
3.3 Adding a point to a configuration ........................................ 52

5.1 An illustration of a quasi-fibration sequence .............................. 107
5.2 An $\varepsilon$-ball with its orbits on an orbifold ............................ 110

6.1 The hyperplane arrangement corresponding to $P\mathbb{S}^2$ .................. 131
6.2 The facet representatives corresponding to $P\mathbb{S}^2$ ....................... 131
6.3 The facet ordering corresponding to $P\mathbb{S}^2$ ........................... 132
6.4 $C_F$ representatives when $F = (1, 1, 1)$ ........................................ 133
6.5 $C_F$ representatives when $F$ is codimension 1 ............................. 133
6.6 $C_F$ representative when $F$ is a chamber .................................. 134
Chapter 1

Introduction

Figure 1.1: Fielders and their ghost fielders on a cricket field. The picture on the left is the original field positions (a configuration on a disc). The picture in the middle adds in the ghost fielders mirroring the original fielders by a reflection in the centre line (a configuration on the orbispace given by a disc with a $\mathbb{Z}_2$ reflection action). The picture on the right has the ghost fielders mirroring the original fielders by a rotation of $\pi$ (a configuration on the orbispace given by a disc with a $\mathbb{Z}_2$ rotation action).

Configuration spaces, being made of $n$ distinct points in a space, appear in a wide variety of disciplines. Consider from thermodynamics, the configuration of $N$ gas molecules in a box. Consider from biology, the configuration of bees in a hive, where bees are allowed to be added to or removed from the system at the entrance of the hive. Consider from sport, the configuration of fielders on a cricket field, where no two can stand at the same point. Imagine further that there is another set of fielders, these ones wearing black, which mirror (or ghost) the moves of the fielders in white, further restricting the original fielders’ movements. What can be said about the space of all such configurations of fielders? What happens when a fielder wearing white runs straight at their ghost and they collide? We have now stepped into the world of configurations
on an orbispace and the idea of ‘ghost points’. See Figure \[\text{fig:ghosts}\] for an illustration. This will be a good example to keep in mind throughout the thesis.

This thesis was born of the idea that homological stability results on manifolds should be able to be extended to the more general case of orbifolds. The standard approach on the interior of a manifold with boundary involves picking a point $\varepsilon \in \partial M$. With this point fixed, one defines a map on configuration spaces of $M$,

$$s_\varepsilon : \text{Conf}_n M \to \text{Conf}_{n+1} M,$$

which adds a point on the boundary of $\overline{M}$ followed by pushing all the configuration points away from the boundary (so that the new point is now in the interior of $\overline{M}$). Results of [Seg73, McD75, Seg79] show that this map, $s_\varepsilon$, induces an isomorphism on integral homology in degrees $* \leq \frac{g}{2}$. Also see [BCT89] for general calculations of the Betti numbers of configuration spaces of manifolds with co-efficients in a field.

This thesis presents ideas on three main areas,

1. homological injectivity in relation to configurations on orbifolds (Chapter 3);
2. a quasi-fibration criterion on orbifolds (Chapter 4); and
3. a construction of the Salvetti complex of the pure braid group suitable for computer programming (Chapter 6).

The foundations are lain in Chapter 2, where both the classical and modern definitions of orbifolds are presented, along with some basic theorems. The classical definition is presented as motivation for the modern definition and to give the reader a feel of how to think of these objects. Without getting overwhelmed with details, a stop-gap definition of an orbifold, $\mathcal{X}$, is a topological category with an object space, $X_0$, and a space of arrows between these points, $X_1$. Extra assumptions include, but are not limited to:

1. $X_0$ and $X_1$ are manifolds;
2. the source and target maps $s, t : X_1 \to X_0$ are local homeomorphisms; and
3. every arrow is invertible.

\(^1\)The author realises that ghosts usually prefer to wear white. Unfortunately for them, the original fielders (being cricketers) were already wearing whites. To reduce confusion, the ghosts were forced to settle for black.
In practice, working with orbifolds is the same as working with a more specific version of topological groupoids. One useful way to create an orbifold is by taking a manifold, \(M\), and using a group, \(G\), which acts on \(M\) to define the arrows, this orbifold is written \([M/G]\).

A configuration on an orbifold is similar to that of a manifold, or more generally, any topological space. The difference here is that the restriction that no two points may be equal is extended so that no two points may be connected by an arrow. In the case of \([M/G]\) from above, this definition gives the same result as taking the orbit configuration space.

The latter part of Chapter 2 investigates mappings between orbifolds. This is achieved by looking at objects known as \(\text{Orb-spaces} (\text{GH07})\), and running through some example calculations.

The goal of Chapter 3 is to prove a homological injectivity result on certain orbifolds. The orbifold in question must be the interior of an orbifold with boundary, which has a neighbourhood of the boundary that is homeomorphic to the boundary crossed with the interval. This assumption gives a place on the orbifold where a point can be added to a configuration on that orbifold. Specifically, the process is to push configuration points away from the boundary, this leaves an empty area to add a new point (this is the orbifold analogue of the \(s_c\) map from earlier). The result of the chapter on homological injectivity is Theorem 3.5.1, it is stated below in a slightly simplified version.

**Theorem 1.** Let \(\mathcal{X}\) be an orbifold which is the interior of an orbifold with boundary of dimension at least 2. Assume that the closure, \(\overline{\mathcal{X}}\), is homeomorphic to \(\partial \mathcal{X} \times I\) near the boundary. Then there exists a map which adds a point near the boundary of \(\overline{\mathcal{X}}\) to a configuration,

\[i : \text{Conf}_{n}\mathcal{X} \rightarrow \text{Conf}_{n+1}\mathcal{X},\]

such that the induced map on integral homology is injective,

\[i_* : H_*(\text{Conf}_n\mathcal{X}) \hookrightarrow H_*(\text{Conf}_{n+1}\mathcal{X}).\]

A natural question to ask, once homological injectivity has been shown, is whether these configurations exhibit any homological surjectivity, or if a stabilising homology can be found. Standard techniques, such as those in \([\text{McD75, RW11, Sal01}]\), fall apart when one moves from the world of manifolds to that of orbifolds. One of the problems with the new setting requires finding a quasi-fibration criterion for orbifolds (or more generally, topological groupoids), which is the subject of Chapter 4.

Recall that a map \(f : A \rightarrow B\) is a quasi-fibration if whenever \(y \in B\) then
the canonical map between the fibre over \( y \) and the homotopy fibre over \( y \) is a homotopy equivalence. That is, the map

\[
f^{-1}(y) \to \hofib(f) = \{(x, \gamma) \mid x \in X, \gamma : I \to B, \gamma(0) = f(x), \gamma(1) = y\},
\]

\( z \mapsto (z, c_y) \),

where \( c_y \) is the constant path at \( y \), is a homotopy equivalence.

The main result of this chapter on quasi-fibrations requires some technical assumptions. Assume a map of topological groupoids, \( f : C \to C' \), satisfies the following,

- if \( h : Z_0 \leftarrow Z_1 \) is an arrow in \( C' \), then there exists a continuous splitting of the target map \( t : E \to H \), where \( E \subseteq C_1 \) is the subspace of arrows \( \{g \mid f(g) = h\} \), and \( H \subseteq C_0 \) is the subspace of objects \( \{X \mid f(X) = Z_0\} \); and

- the map \( (Bf)^{-1}(C'_0) \xrightarrow{Bf} C'_0 \), the map from the fibre over the objects to the objects, is a quasi-fibration. In this condition, \( Bf \) is the induced map on classifying spaces \( BC \to BC' \).

In this case, the following theorem is true and appears as Theorem 4.3.1.

**Theorem 2.** Assuming the above conditions, the entire map

\[
Bf : BC \to BC',
\]

is a quasi-fibration.

This theorem looks complicated, and doesn’t seem as if it would be overly pleasant to use. The good news is, however, that using the above theorem as a base, the chapter finishes with a Theorem 4.4.1.

**Theorem 3.** Let \( \mathcal{M} \) be a connected \( n \)-orbifold with compact coarse space. Let \( \mathcal{M}' \subseteq \mathcal{M} \) be a full \( n \)-suborbifold with compact coarse space, and let \( \mathcal{N} \) be a full closed \( n \)-suborbifold such that \( (\mathcal{M}', \mathcal{N} \cap \mathcal{M}') \) is connected. Then there is a quasi-fibration sequence

\[
B\text{Conf}(\mathcal{M}', \mathcal{N} \cap \mathcal{M}') \to B\text{Conf}(\mathcal{M}, \mathcal{N}) \xrightarrow{Bf} B\text{Conf}(\mathcal{M}, \mathcal{M}' \cup \mathcal{N}).
\]

In this theorem, the configuration spaces \( \text{Conf}(\mathcal{M}, \mathcal{N}) \) are configurations of points in \( \mathcal{M} \) where points are allowed to be added or removed in \( \mathcal{N} \). The map \( f \) simply increases the area where points can be added or removed. This theorem is one that could prove very useful in a future proof finding the stable homology on configurations of orbifolds.
This work is followed by a short chapter on finding a homotopy between configurations on an orbifold and the orbifold of sections over that orbifold. The manifold case appears in [McD75] and reads:

**Theorem 4 ([McD75]).** Let $X$ be a manifold with boundary $\partial X = L \cup L'$ such that $\dim(L) = \dim(L') = \dim(\partial X)$ and $\partial L = \partial L' = L \cap L'$. Then if $X$ is connected and $L$ is non-empty, there is a homotopy equivalence,

$$Conf(X,L) \rightarrow \Gamma(X,L').$$

The space on the left is configurations on $X$ where points can be added or forgotten in $L$ and the space on the right is certain sections over $X$ which are trivial over $L'$. The specific definition of this section space is given in Chapter 5, along with some work towards getting a similar homotopy equivalence in the orbifold setting. So that expectations are not too high: this work is not complete.

In Chapter 6, the final chapter of this thesis, the focus turns to concrete calculations. Viewing the ordered configuration space on $C$ as the compliment of a hyperplane arrangement,

$$PConf_n C = \mathbb{C}^n \setminus \bigcup_{i \neq j} \{(x_1, \ldots, x_n) \in \mathbb{C}^n | x_i = x_j\},$$

one is able to construct its Salvetti complex [Sal87]. The Salvetti complex has a simplicial complex description which can be found in [Par93], but is also presented in Chapter 6 for convenience. The chapter then discusses a way to view the Salvetti complex construction in a way which can be coded into a computer. Some of the algorithms are presented, and reference computer code can be found in Appendix A.

As a word of caution, the phasmophobic reader is advised to skip Section 2.3.
Chapter 2

Orbifold Background

This chapter begins with the definitions of orbifolds and orbispaces, in both the classical and groupoid approaches (Section 2.1). In Section 2.2, the analogue of mapping spaces in the world of orbispaces is introduced, along with Orb-spaces, before the analogue of configuration spaces for orbispaces is presented in Section 2.3. The chapter concludes with some examples with the functor which takes topological groupoids to Orb-spaces in Section 2.4.

2.1 Definitions and Preliminary Examples

This section gives some background on orbispaces and orbifolds. It begins with the definition of a classical orbifold, objects which look locally like a space quotiented by a group. The definition of a modern orbifold, which involves groupoids, is then presented. A reference for background on orbifolds is [ALR07], though a slightly different definition of an orbifold will be used, one much closer to that of [GH07]. The differences will be noted when they are encountered.

2.1.1 Classical Orbifolds

The definition of a classical orbifold gives an intuitive feel for what an orbifold looks like and is good motivation for the later, more generalised, definition.

Compare a classical orbifold to the idea of a manifold. A manifold can be thought of as a topological space which looks like \( \mathbb{R}^n \) at each point. One can then think of a classical orbifold as being an object which, at each point \( x \), looks like the quotient space \( \mathbb{R}^n/G_x \) for finite groups \( G_x \). The technical definition as given in [ALR07] follows.

**Definition 2.1.1.** Let \( X \) be a topological space and set the dimension \( n \geq 0 \).
An $n$-dimensional (smooth) orbifold chart on $X$ is a tuple $(\tilde{U}, G, \varphi)$ where

- $\tilde{U} \subseteq \mathbb{R}^n$ is a connected open subset;
- $G$ is a finite group of smooth automorphisms of $\tilde{U}$; and
- $\varphi : \tilde{U} \to X$ is a $G$-invariant map such that $\varphi$ induces a homeomorphism of $\tilde{U}/G$ onto an open subset $U \subseteq X$.

An embedding, $\lambda : (\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$ between two orbifold charts is a smooth embedding $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ such that $\psi \circ \lambda = \varphi$.

An orbifold atlas on $X$ is a family $U$ of orbifold charts on $X$ such that they cover $X$ and are locally compatible. That is, if $(\tilde{U}, G, \varphi), (\tilde{V}, H, \psi) \in U$ are two orbifold charts (for $U \cong \tilde{U}/G$ and $V \cong \tilde{V}/H$), and $x \in U \cap V$, then there exists an open neighbourhood of $x$, $W \subseteq U \cap V$, and a chart, $(\tilde{W}, K, \mu)$, for $W$, such that there are two embeddings,

$$
(\tilde{W}, K, \mu) \hookrightarrow (\tilde{U}, G, \varphi),
(\tilde{W}, K, \mu) \hookrightarrow (\tilde{V}, H, \psi).
$$

An orbifold atlas $U$ is said to refine another orbifold atlas $V$ if whenever $(\tilde{U}, G, \varphi) \in U$ then there exists an orbifold chart $(\tilde{V}, H, \psi) \in V$ such that there is an embedding $(\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$.

Two orbifold atlases $U$ and $V$ are equivalent if there exists a third orbifold atlas $W$ such that $W$ refines both $U$ and $V$.

**Definition 2.1.2.** A classical $n$-orbifold $X$ consists of

1. a paracompact Hausdorff space, $X$; and
2. an equivalence class of $n$-dimensional orbifold atlases on $X$, $[U]$.

The simplest example is the trivial orbifold structure on a regular $n$-manifold.

**Example 2.1.3.** Let $M$ be an $n$-dimensional manifold with manifold atlas $\mathcal{U} = \{ (\tilde{U}, \varphi) \}$. That is, each $\tilde{U} \subseteq \mathbb{R}^n$ and $\varphi : \tilde{U} \to M$ is a homeomorphism onto an open subset $U \subseteq M$, along with local compatibility conditions. Then a trivial orbifold atlas on $M$ is the atlas $\mathcal{U}' = \{ (\tilde{U}, \{1\}, \varphi) \}$ where $\{1\}$ is the trivial group which acts as the identity on each $\tilde{U}$.

**Example 2.1.4.** Let $D^2$ be the unit disc and let $\frac{\mathbb{Z}}{2}$ be the cyclic group on two elements. The orbifold written $D^2/\frac{\mathbb{Z}}{2}$ (with $\frac{\mathbb{Z}}{2}$ acting by reflection) is thought of as the 2-disc folded in half, or a half disc with a reflecting plane through the origin. A manifold atlas for $D^2$ is the set of one element $\mathcal{U} = \{ (D^2 \subseteq \mathbb{R}^2, id) \}$. 

Let \( X = \{ (x, y) \in D^2 \mid y \geq 0 \} \), the top half disc, define \( U' \), an orbifold atlas on \( X \) for \( D^2/\mathbb{Z}_2 \), by setting

\[
U' = \{ (D^2, \mathbb{Z}_2, \nu) \},
\]

where

\[
\nu : D^2 \to X,
\]

\[
(x, y) \mapsto (x, |y|),
\]

and the \( \mathbb{Z}_2 \) in the second factor of \( U' \) acts by reflection on \( D^2 \). See Figure 2.1.

It is important to keep in mind that this reflection action of \( \mathbb{Z}_2 \) on \( D^2 \) is not the only way that \( \mathbb{Z}_2 \) can act on \( D^2 \). Another possibility is the rotation by \( \pi \) action (multiplication by \( e^{i\pi} \) on \( D^2 \subseteq \mathbb{C} \)), the corresponding orbifold is also written \( D^2/\mathbb{Z}_2 \). Therefore, one should take care to explicitly state whether \( \mathbb{Z}_2 \) is acting by reflection or rotation on \( D^2 \) when writing \( D^2/\mathbb{Z}_2 \). This ambiguity disappears when looking at \( \mathbb{Z}_n \) acting on \( D^2 \) for \( n \geq 3 \), as in these cases there is no possible reflection with a \( \mathbb{Z}_n \) action. In the following example of \( \mathbb{Z}_n \) acting on \( D^2 \) by rotation for \( n = 2 \), the higher \( n \) values are obtained by the obvious extension.

**Example 2.1.5.** The orbifold written \( D^2/\mathbb{Z}_2 \) (with \( \mathbb{Z}_2 \) acting by rotation) can be thought of as a half disc with one half of the flat boundary glued to the other half of the flat boundary. Use the same manifold atlas for \( D^2 \) as in the previous example: \( U = \{ (D^2 \subseteq \mathbb{C}, id) \} \). Let \( X \) be the unit disc in \( \mathbb{C} \). Define \( U' \), an orbifold atlas on \( X \) for \( D^2/\mathbb{Z}_2 \), by setting

\[
U' = \{ (D^2, \mathbb{Z}_2, \nu) \},
\]
Figure 2.2: The classical orbifold $D^2/\mathbb{Z}_2$ where $\frac{\pi}{2}$ acts by rotation by $\pi$. The centre point on the right-hand-side is the only point with $\frac{\pi}{2}$ isotropy. The left side of the centre line is identified with the right side of the centre line by reflection.

Figure 2.3: The teardrop orbifold is $S^2$ with a single point of non-trivial isotropy. In this case the north-pole has $\frac{\pi}{2}$ isotropy.

where

$$\nu : D^2 \to X,$$
$$re^{i\theta} \mapsto re^{i2\theta},$$

and the $\frac{\pi}{2}$ in the second factor of $U'$ acts as rotation by $\pi$ on $D^2$. See Figure 2.3.

Example 2.1.6. While previous examples were given as the quotient of a manifold by a finite group, this example cannot be expressed in such a way. The teardrop orbifold is constructed from $S^2$ with a single point of non-trivial isotropy (See Figure 2.3). The north-pole in the example has $\frac{\pi}{2}$ isotropy, the remaining points have trivial isotropy.

This example can be generalised to the spindle orbifold. Given $n, m \in \mathbb{Z}_{>0}$,
2.1 Definitions and Preliminary Examples

If \( n \neq m \), one can form a spindle orbifold by taking an \( S^2 \) whose north pole has \( \mathbb{Z}_n \) isotropy and whose south pole has \( \mathbb{Z}_m \) isotropy. The charts on the spindle orbifold are:

- the chart for \( D^2/\mathbb{Z}_n \) on the northern hemisphere; and
- the chart for \( D^2/\mathbb{Z}_m \) on the southern hemisphere.

These two charts are glued together along their boundary cylinders.

2.1.2 The Groupoid Approach to Orbifolds

Though the classical definition gives a good feel for what an orbifold is, a more general definition uses groupoids. The intuition for this definition is to think of an orbifold as a manifold with an added arrow space of invertible morphisms. Two points are thought of as being ‘sort of the same’ if they are connected by an arrow. Taking the quotient of the manifold by the arrow space gives what is known as the coarse space of the orbifold, an object closely related to a classical orbifold.

Definition 2.1.7. A groupoid is a category in which every morphism is invertible.

Begin with the idea of an orbispace, which is closely related to topological groupoids. Notionally, orbispaces are to topological spaces as orbifolds are to manifolds.

Definition 2.1.8. A topological groupoid \( \mathcal{G} \) is a groupoid object in the category of topological spaces. Explicitly, a topological groupoid \( \mathcal{G} \) has an object space \( \mathcal{G}_0 \) and an arrow space \( \mathcal{G}_1 \) with five continuous structure maps:

- The source map, \( s : \mathcal{G}_1 \to \mathcal{G}_0 \), and target map, \( t : \mathcal{G}_1 \to \mathcal{G}_0 \), which give the source and target of an arrow, \( s(g) \) and \( t(g) \) respectively. If \( x, y \in \mathcal{G}_0 \) then one writes \( g : x \to y \) or \( x \xrightarrow{g} y \) to say that \( g \in \mathcal{G}_1 \) with \( s(g) = x \) and \( t(g) = y \).

- The composition map \( m : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1 \). Let \( g, h \in \mathcal{G}_1 \) such that \( s(h) = t(g) \), the composition of these two arrows, \( hg \in \mathcal{G}_1 \), should satisfy \( s(hg) = s(g) \) and \( t(hg) = t(h) \). To define this arrow, if \( g : x \to y \) and \( h : y \to z \), then the composition \( hg \) is defined and \( hg : x \to z \). The composition map is defined by \( m(h, g) = hg \) on the fibred product,

\[
\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 = \{(h, g) \in \mathcal{G}_1 \times \mathcal{G}_1 \mid s(h) = t(g)\},
\]

such that composition is associative.
• The unit map or identity map $u : G_0 \to G_1$, is a two-sided identity for composition. That is, if $x, y \in G_0$ and $g : x \to y$ then $s(u(x)) = t(u(x)) = x$ and $gu(x) = g = u(y)g$.

• An inverse map $i : G_1 \to G_1$ (often written $i(g) = g^{-1}$). If $g : x \to y$ then $g^{-1} : y \to x$, $g^{-1}g = u(x)$ and $gg^{-1} = u(y)$.

Example 2.1.9. A trivial example occurs when $X$ is a topological space. One can construct a topological groupoid from $X$ by simply setting $G_0 = X$ and arrow space $G_1 = \{id : x \to x | x \in G_0\} = X$.

Example 2.1.10. Let $G$ be a topological group. The topological groupoid on one object with arrows from $G$, written $[\ast / G]$ is the groupoid with

• object space $\{\ast\}$; and

• arrow space $\{g : \ast \to \ast | g \in G\}$.

The composition map comes from multiplication in $G$. The identity map at $\ast$ is the identity element in the group, $u(\ast) = 1_G$. The inverse of an arrow is given by the inverse of the group element in $G$.

Example 2.1.11. This idea of $[\ast / G]$ can be extended to $[M / G]$ for $M$ a manifold. When $G$ is a topological group which acts on $M$, the topological groupoid $[M / G]$ is the groupoid with

• object space $M$; and

• arrow space $\{(g, m) : m \to g(m) | g \in G, m \in M\}$.

Definition 2.1.12. Let $\mathcal{G}$ be a topological groupoid. The isotropy of a point $x \in G_0$ is the set

$$\{g \in G_1 | s(g) = t(g) = x\}.$$ 

Definition 2.1.13. Let $\mathcal{G}$ and $\mathcal{H}$ be two topological groupoids. A topological groupoid homomorphism, $\varphi : \mathcal{H} \to \mathcal{G}$, is a pair of continuous maps,

$$\varphi_0 : H_0 \to G_0, \quad \varphi_1 : H_1 \to G_1,$$

which commute with all the structure maps of $\mathcal{G}$ and $\mathcal{H}$.

Definition 2.1.14. A Morita equivalence between topological groupoids $\mathcal{H}$ and $\mathcal{G}$ is a homomorphism $\varphi : \mathcal{H} \to \mathcal{G}$ such that
1. the map
\[ t \circ \text{proj}_1 : G_1 \times \varphi H_0 \to G_0 \]
\[ (g, y) \mapsto t(g), \]
defined on \( \{(g, y) \mid g \in G_1, y \in H_0, s(g) = \varphi(y)\} \) is a surjective local homeomorphism; and

2. the square
\[
\begin{array}{ccc}
H_1 & \xrightarrow{\varphi} & G_1 \\
\downarrow (s, t) & & \downarrow (s, t) \\
H_0 \times H_0 & \xrightarrow{\varphi \times \varphi} & G_0 \times G_0
\end{array}
\]
is a fibred product.

The first condition is the analogue of essential surjectivity from regular category theory as it implies that every object in \( \mathcal{H} \) can be connected to the image of \( \varphi \) by an arrow in \( \mathcal{G} \). The second condition is the analogue of a functor being full and faithful as it implies that \( \varphi \) induces a homeomorphism
\[ H_1(y, z) \to G_1(\varphi(y), \varphi(z)), \]
where \( H_1(y, z) \) is the space of all arrows in \( H_1 \) from \( y \) to \( z \), similarly for the right-hand-side.

The existence of a Morita equivalence is not enough to define an equivalence relation between topological groupoids. To get this equivalence relation, the following is required.

**Definition 2.1.15.** Two topological groupoids \( \mathcal{G} \) and \( \mathcal{G}' \) are called *Morita equivalent* if there exists a topological groupoid \( \mathcal{H} \) and Morita equivalences:
\[ \varphi : \mathcal{H} \to \mathcal{G}, \]
\[ \varphi' : \mathcal{H} \to \mathcal{G}'. \]

This defines an equivalence relation on topological groupoids. Morita equivalent topological groupoids preserve certain properties. For example under Morita equivalence, properness (Definition \[\text{[2.1.20]}\]) is invariant, whereas étaleness (Definition \[\text{[2.1.27]}\]) is not necessarily invariant. See [ALR07] for more details.

**Definition 2.1.16.** Let \( \mathcal{G} \) be a topological groupoid. The *coarse* (sometimes:
orbit) space of $G$, denoted $|G|$ is defined to be the quotient

$$|G| := G_0/\sim,$$

where $x \sim y$ if there is an arrow $x \to y$ in $G_1$.

**Definition 2.1.17.** An orbispace structure on a paracompact Hausdorff space $X$ consists of a topological groupoid $G$ and a homeomorphism $f : |G| \to X$.

An orbispace, $X$, is a topological space, $X$, equipped with an orbispace structure.

In most cases, when writing ‘let $X$ be an orbispace’, the author will work directly with the corresponding topological groupoid $G$ and, through abuse of notation, write $X_0 := G_0$ and $X_1 := G_1$.

This is the definition of orbispace used in the sequel. A word of caution, this definition disagrees with the definition of orbifold in [ALR07], among other places, in which orbifolds are defined having an equivalence class of orbifold structures. The definition above is the one used in [GH07] and, as mentioned within, allows the definition of mapping orbispaces, something which is not possible with the equivalence class definition. The equivalence class version of the definition will be presented below for reference.

**Definition 2.1.18.** If $\varphi : \mathcal{H} \to \mathcal{G}$ is a Morita equivalence, then $|\varphi| : |\mathcal{H}| \to |\mathcal{G}|$ is a homeomorphism, and the composition $f \circ |\varphi| : |\mathcal{H}| \to X$ is said to define an equivalent orbispace structure on $X$.

Note that if two topological groupoids $G$ and $G'$ are Morita equivalent and $G$ represents an orbispace structure on $X$, then $G'$ represents an equivalent orbispace structure on $X$ (with a suitable $f'$).

**Remark 2.1.19** (Equivalence class version of orbispaces). In the equivalence class version of orbispaces (see [ALR07]), an orbispace, $X$, is defined to be a space, $X$, equipped with an equivalence class of orbispace structures.

In this case, a presentation of $X$ is an element of the equivalence class. That is, a choice of topological groupoid, $G$, and a homeomorphism $f : |G| \to X$.

**Example 2.1.20.** The first example of an orbispace is $[*/G]$, for a topological group $G$. Note that this is the same notation used for the topological groupoid coming from a topological group $G$ in Example 2.1.10. The reason for this is that the two objects are very closely related allowing for very little confusion coming from this notation collision.

The orbispace $[*/G]$ is the point $*$, equipped with the orbispace structure groupoid $[*/G]$ (Example 2.1.10) and homeomorphism $\varphi : |[*/G]| = * \to *$. 
2.1. DEFINITIONS AND PRELIMINARY EXAMPLES

There are similar examples of orbispaces to those that were presented for classical orbifolds.

**Example 2.1.21.** This is the orbispace analogue of Example 2.1.9. Let $M$ be a manifold. Let $\mathcal{G}$ be the trivial topological groupoid coming from $M$ (see Example 2.1.9), that is, $\mathcal{G}$ has

- object space $M$; and
- morphism space $\{id : x \to x \mid x \in M\} = M$.

Then the trivial orbispace structure on $M$ is the structure defined by the topological groupoid $\mathcal{G}$ and homeomorphism $id : |\mathcal{G}| = M \to M$.

**Example 2.1.22.** This is the orbispace analogue of Example 2.1.4. Recall that this example involved the disc with $\mathbb{Z}_2$ acting by reflection over a plane through the origin. Let $D^2$ be the unit disc in $\mathbb{R}^2$, and let $D^2_+ = \{(x, y) \in D^2 \mid y \geq 0\}$. Let $\mathcal{G}$ be the topological groupoid with

- object space $D^2$; and
- morphism space generated by $\{(x, y) \to (x, -y) \mid (x, y) \in D^2\}$.

The orbispace $[D^2/\mathbb{Z}_2]$ with $\mathbb{Z}_2$ acting by reflection on $D^2$ is the structure defined by the topological groupoid $\mathcal{G}$ and the obvious homeomorphism $|\mathcal{G}| \cong D^2_+$. That is, the map $\varphi : |\mathcal{G}| \to D^2_+$ defined by forcing that the composition

$$D^2_+ \hookrightarrow \mathcal{G} \to |\mathcal{G}| \xrightarrow{\varphi} D^2_+,$$

is the identity, where the first map is inclusion into the object space of $\mathcal{G}$.

**Example 2.1.23.** This is the orbispace analogue of Example 2.1.5. Recall that this example involved the disc with $\mathbb{Z}_2$ acting by a rotation of $\pi$. This example will go one step further and present the general $\mathbb{Z}_n$ rotation case, where $n \in \mathbb{Z}_{\geq 1}$. Let $D^2$ be the unit disc in $\mathbb{C}$. Let $\mathcal{G}$ be the topological groupoid with

- object space $D^2$; and
- morphism space generated by $\{x \to e^{2\pi i/n}x \mid x \in D^2\}$.

The orbispace $[D^2/\mathbb{Z}_n]$, with $\mathbb{Z}_n$ acting by a rotation of $\frac{2\pi}{n}$, is the structure defined by the topological groupoid $\mathcal{G}$ and the homeomorphism $|\mathcal{G}| \cong D^2$ induced by the surjective map

$$\varphi : G_0 \to D^2, \quad re^{i\theta} \mapsto re^{in\theta}.$$  

\footnote{To say that a morphism space is generated by a set, this means adding in identity morphisms, composition morphisms and everything else needed to make $\mathcal{G}$ a topological groupoid.}
When comparing these examples to the classical orbifold cases (Examples 2.1.4 and 2.1.5), the similarities and subtle differences should be noted. In the classical examples, the group actions of $\mathbb{Z}_2$ played an important role in the definition (when finding an orbifold atlas with $\mathbb{Z}_2$ invariant maps). In the orbispace examples, these groups become obscured by the topological groupoid machinery. However, one should observe the similarity between the $\mathbb{Z}_2$ invariant maps in the classical orbifold atlas, and the map

$$G_0 \to |G|,$$

in the orbispace case.

These two examples, above, of the different orbispaces $\left[ D^2 / \mathbb{Z}_2 \right]$ also happen to be examples of modern orbifolds, the definition of which follows.

**Definition 2.1.24.** A groupoid $\mathcal{G}$ is called proper if $(s,t) : G_1 \to G_0 \times G_0$ is a proper map. That is, preimages of compact subsets of $G_0 \times G_0$ are compact in $G_1$.

Both $G_0$ and $G_1$ are assumed to be Hausdorff.

**Definition 2.1.25.** A modern orbifold (in the sequel: orbifold), is an orbispace $\mathcal{X}$ such that

1. both $s$ and $t$ are local homeomorphisms;
2. the structure groupoid of $\mathcal{X}$ is proper; and
3. both the object space and the arrow space of the structure groupoid of $\mathcal{X}$ are manifolds.

Even ignoring the lack of equivalence class of orbispace structures, this definition still differs to that found in [ALR07]. In this book, the word orbifold is reserved for objects with extra smoothness assumptions. Such objects in this document are referred to as smooth orbifolds, the definition of which follows.

**Definition 2.1.26.** A topological groupoid $\mathcal{G}$ is called a Lie groupoid if

1. the object and morphism spaces $G_0$ and $G_1$ are smooth manifolds;
2. the structure maps $s$ and $t$ are smooth submersions; and
3. the structure maps, $m$, $u$ and $i$, are smooth.

**Definition 2.1.27.** A Lie groupoid $\mathcal{G}$ is called étale if both $s$ and $t$ are local diffeomorphisms.
Definition 2.1.28. A smooth orbifold is an orbispace $X$ such that the structure groupoid of $X$ is a proper étale Lie groupoid.

Note here that when working with smooth orbifolds, a slightly different definition of Morita equivalence (Definition 2.1.14) should be used. In this case, the first condition should instead require that $t \circ \text{proj}_1 : G_1 \times \phi H_0 \to G_0$ is a surjective local diffeomorphism.

Taking the equivalence class version of this definition for smooth orbifolds, as was done with orbispaces (see Remark 2.1.19), one obtains the definition of an orbifold as given in [ALR07].

It is also useful for the later work to define an orbifold with boundary, the conditions here are the logical extension of the ones in Definition 2.1.25.

Definition 2.1.29. An orbifold with boundary is an orbispace $X$ such that

1. both $s$ and $t$ are local homeomorphisms;
2. the structure groupoid of $X$ is proper; and
3. both the object space and the arrow space of the structure groupoid of $X$ are manifolds with boundary.

Note that putting conditions 1 and 3 together, an arrow $x \to y$ in $X$ with $x \in \partial X_0$ implies that $y \in \partial X_0$.

2.2 An Analogue of Mapping Spaces for Orbispaces

This section works to define an orbispace analogue of mapping spaces in the same approach as that used in [GRU07].

Definition 2.2.1. Let $\mathcal{G}$ and $\mathcal{H}$ be orbispaces. Write $\text{map}(\mathcal{H}, \mathcal{G})$ for the space of continuous functors from $\mathcal{H}$ to $\mathcal{G}$.

Furthermore, write $\text{Map}(\mathcal{H}, \mathcal{G})$ for the category with

- object space $\text{map}(\mathcal{H}, \mathcal{G})$; and
- morphism space being continuous natural transformations.

For the definition of $\text{map}(\mathcal{H}, \mathcal{G})$, note that the space of continuous functors is a subset of $\text{map}(H_0, G_0) \times \text{map}(H_1, G_1)$ with the compact-open topology. Similarly for the topology on the arrow space of $\text{Map}(\mathcal{H}, \mathcal{G})$, a natural transformation assigns to each object in $\mathcal{H}$ an arrow in $\mathcal{G}$. A natural transformation is therefore a map $\eta : H_0 \to G_1$. The space of all natural transformations is a subset of $\text{map}(H_0, G_1)$ with the compact-open topology. The topologies on these sets are used to define the continuity.
Example 2.2.2. A simple example has $\mathcal{H} = */H$ and $\mathcal{G} = */G$ where $G$ and $H$ are topological groups. The space $\text{map}(*/H, */G)$ is the space of continuous functors, that is, every element is made up of

- a map (for the objects) $* \rightarrow *$; and
- a map (for the arrows) $H \rightarrow G$;

which commute with all the structure maps. This means that $\text{map}(*/H, */G)$ is simply the group homomorphisms $\text{Hom}(H, G)$.

This is also the object space of $\text{Map}(*/H, */G)$. The morphism space of this category is made up of all natural transformations between the elements of $\text{map}(*/H, */G)$. Let $F, F' \in \text{map}(*/H, */G)$. A natural transformation $\eta : F \Rightarrow F'$ is made up of:

1. A morphism from $*/G$ for each $x \in \text{obj}(*/H)$, written $\eta_x : F(x) \rightarrow F'(x)$. In this setting, note that the only object is $*$, so all the morphisms are of the form $\eta_x : * \rightarrow *$ coming from $*/G$. Thus, $\eta_x$ is simply an element of $G$.

2. For every morphism $\varphi : * \rightarrow *$ in $*/H$, a commutative diagram

\[
\begin{array}{ccc}
* & \xrightarrow{F(\varphi)} & * \\
\downarrow{\eta_x} & & \downarrow{\eta_x} \\
* & \xrightarrow{F'(\varphi)} & *
\end{array}
\]

in the category $*/G$. This is simply the requirement that $F'(\varphi) \circ \eta_x = \eta_x \circ F(\varphi)$, or equivalently: $F'(\varphi) = \eta_x \circ F(\varphi) \circ \eta_x^{-1}$. That is, an arrow from $F$ to $F'$ is an element of $G$ which conjugates $F(\varphi)$ to $F'(\varphi)$ for every $\varphi \in H$.

This category can be written

$$\text{Map}(*/H, */G) = [\text{Hom}(H, G)/G],$$

where the action of $G$ is given by

$$h \cdot f(x) = hf(x)h^{-1}.$$  

In the above equation, $h$ acts on the left-hand-side by the action in the groupoid, $h$ acts on the right-hand-side by the action in the group $G$.  


2.2.1 Orb-spaces

At this point, it is a good time to introduce the idea of the fat geometric realisation, a way to think of categories as topological spaces. Geometric realisations are a useful tool in a range of applications, these include defining classifying spaces of categories, and defining homotopy limits and colimits.

Definition 2.2.3 (Fat Geometric Realisation). Let \( \mathcal{G} \) be a category. Define,

\[
G_n := \{ x_0 \xleftarrow{\alpha_1} x_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_n} x_n | \text{composable morphisms in } \mathcal{G} \},
\]

\[
\Delta^n := \{ (r_0, \ldots, r_n) | r_i \geq 0, \sum r_i = 1 \}.
\]

The fat geometric realisation of \( \mathcal{G} \), written \( ||\mathcal{G}|| \), is the topological space,

\[
||\mathcal{G}|| := \bigsqcup_{n \geq 0} G_n \times \Delta^n / \sim,
\]

where \( \sim \) is the equivalence relation generated by:

1. \( 0 \cdot x_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_n} r_n \cdot x_n \sim r_1 \cdot x_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_n} r_n \cdot x_n; \)

2. \( r_0 \cdot x_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_n} 0 \cdot x_n \sim r_0 \cdot x_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_{n-1}} r_{n-1} \cdot x_{n-1}; \) and

3. \( r_0 \cdot x_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_{i-1}} r_{i-1} \cdot x_{i-1} \xleftarrow{\alpha_i} 0 \cdot x_i \xleftarrow{\alpha_{i+1}} \cdots \xleftarrow{\alpha_n} r_n \cdot x_n \sim r_0 \cdot x_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_{i-1}} r_{i-1} \cdot x_{i-1} \xleftarrow{\alpha_i \circ \alpha_{i+1}} \cdots \xleftarrow{\alpha_n} r_n \cdot x_n. \)

Intuitively, the equivalence relations can be thought of as allowing terms to be forgotten when the co-efficient of \( x_i \) is zero. If the term with zero co-efficient is in between two arrows, then when it is dropped the surrounding arrows are composed. If the term with zero co-efficient is at either end, then when it is dropped, the single arrow going to or from it is also forgotten.

Note that another way to define the fat classifying space is via the nerve of \( \mathcal{G} \). Such a definition does not appear in this chapter as it is less intuitive for the current purpose. For readers interested in this alternate construction, it is obtained after making a slight modification to the definition of the two-sided bar construction (Definition 4.1.3). Setting \( X \) and \( Y \) to be \( \ast \) in Definition 4.1.3 and ignoring the first equivalence relation gives the fat classifying space defined above (the reason the equivalence relation is ignored is due to wanting the fat version of the construction).

An Orb-space gives an alternate way to think of orbispaces. These categories were originally defined in [GH07]. In this paper, Gepner and Henriques...
introduce model category structures on the categories of topological groupoids and orbispaces and an equivalence between them. The reader is encouraged to consult this reference for a proper introduction to this material. The equivalence will not be reproduced here, as the description of which requires the use of the fibrant replacement functor on topological groupoids, defined below (Definition 2.2.9). The final goal is to use this work to approach the derived mapping spaces between orbispaces, which requires the full force of the homotopy theory developed by Gepner and Henriques.

Orb-spaces are defined relative to an arbitrary family of topological groups, $\mathcal{F}$. This $\mathcal{F}$ should be thought of as the set of allowed isotropy groups in an orbispace. The precise restrictions on the family $\mathcal{F}$ are:

1. $\mathcal{F}$ is closed under isomorphism, and the isomorphism classes of $\mathcal{F}$ are able to be indexed by a set; and

2. if $n \in \mathbb{Z}_{>0}$ and $G \in \mathcal{F}$ then $G^n$ is a paracompact space. That is, every open cover of $G^n$ can be refined to a locally finite open cover.

**Definition 2.2.4.** Let $\mathcal{F}$ be such a family of topological groups. The topologically enriched category $\text{Orb}_\mathcal{F}$ has

- objects being the groups in $\mathcal{F}$; and

- morphisms from $H$ to $G$ being the fat geometric realisation

$$\text{Orb}_\mathcal{F}(H, G) := |\text{Map}([*/H], [*/G])|.$$  

**Definition 2.2.5.** An $\text{Orb}_\mathcal{F}$-space is a continuous contravariant functor from $\text{Orb}_\mathcal{F}$ to spaces.

The upcoming definitions require the idea of left and right $\mathcal{G}$ spaces.

**Definition 2.2.6** ([ALR07] Definition 2.14). Let $\mathcal{G}$ be an orbispace. A left $\mathcal{G}$-space is a topological space $E$ with an action of $\mathcal{G}$. The action comes in two parts,

1. an anchor map $\pi : E \to G_0$, and

2. an action map $\mu : G_1 \times_{G_0} E \to E$, which is defined on $(g, e)$ when $\pi(e) = s(g)$ (also written $\mu(g, e) = g \cdot e$).

The action satisfies the usual identities for an action, $\pi(h \cdot e) = t(h)$, $1_x \cdot e = e$ and $g \cdot (h \cdot e) = (gh) \cdot e$ when $x \overset{h}{\to} y \overset{g}{\to} z$ in $G_1$ and $e \in E$ with $\pi(e) = x$.

A few modifications will allow one to define a right $\mathcal{G}$-space.
2.2. AN ANALOGUE OF MAPPING SPACES FOR ORBISPACES

**Definition 2.2.7.** Let $\mathcal{G}$ be an orbispace. A right $\mathcal{G}$-space is a topological space $E$ with an action of $\mathcal{G}$. The action comes in two parts,

1. an anchor map $\pi : E \to G_0$, and

2. an action map $\mu : E \times_{G_0} G_1 \to E$, which is defined on $(e, g)$ when $\pi(e) = t(g)$ (also written $\mu(e, g) = e \cdot g$).

The action satisfies the usual identities for an action, $\pi(e \cdot g) = s(g)$, $e \cdot 1_z = e$ and $(e \cdot g) \cdot h = e \cdot (gh)$ when $x \xrightarrow{h} y \xrightarrow{g} z$ in $G_1$ and $e \in E$ with $\pi(e) = z$.

The following functor can be used to take a topological groupoid to an $\text{Orb}$-space.

**Definition 2.2.8.** The functor $R$ from topological groupoids to $\text{Orb}$-spaces is defined by

$$R : \{\text{Topological Groupoids}\} \to \{\text{Orb}_x\text{-spaces}\},$$

$$\mathcal{G} \mapsto (G \mapsto \text{map}(\text{fib}([\ast / G]), \text{fib}(G))),$$

where $\text{fib}$ is the fibrant replacement functor.

Examples related to this functor can be found in Section 2.4. The fibrant replacement of $\mathcal{G}$ appears here because it is better suited for receiving maps from other groupoids, while at the same time being Morita equivalent to $\mathcal{G}$.

**Definition 2.2.9.** Let $\mathcal{G}$ be a topological groupoid. The fibrant replacement of $\mathcal{G}$, written $\text{fib}(\mathcal{G})$ is the groupoid

$$\text{fib}(\mathcal{G}) := ((||\mathcal{G} \times G_1|| \times_{G_0} ||\mathcal{G} \times G_1||)/\mathcal{G} \equiv ||\mathcal{G}||),$$

where $\mathcal{G}$ acts by the diagonal action on $||\mathcal{G} \times G_1|| \times_{G_0} ||\mathcal{G} \times G_1||$.

The topological groupoid $\mathcal{G} \times G_1$ is the groupoid with object space $G_1$ and arrows being pairs of composable morphisms in $G_1$. The quotient by $\mathcal{G}$ comes from viewing the two $||\mathcal{G} \times G_1||$ as left and right $\mathcal{G}$-spaces (for the specifics of how, see the proof of the upcoming Claim 2.2.11).

At first sight, the idea of the fibrant replacement may appear opaque. The following work to unpack the definition, along with some examples, should clear this up to some extent. The object space of the fibrant replacement, $||\mathcal{G}||$, is simply the fat geometric realisation of $\mathcal{G}$ (See Definition 2.2.3). As for the arrow space, see the following claim.
Claim 2.2.10. The arrow space of the fibrant replacement, \( (\lvert\mathcal{G} \rtimes G_1\rvert \times_{G_0} \lvert\mathcal{G} \rtimes G_1\rvert) / \mathcal{G} \), is made of objects of the form,

\[
(r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n \leftarrow^p p, \quad \sum r_i = \sum s_i = 1 \text{ and the } \alpha_i, \beta \text{ and } \gamma_i \text{ are composable arrows in } \mathcal{G}.
\]

Proof. Consider the elements of \( (\lvert\mathcal{G} \rtimes G_1\rvert \times_{G_0} \lvert\mathcal{G} \rtimes G_1\rvert) / \mathcal{G} \). The elements of \( \lvert\mathcal{G} \rtimes G_1\rvert \) are similar to those of \( \lvert\mathcal{G}\rvert \), the only difference being that the former has an extra arrow. These elements can be written as either

\[
r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n \leftarrow^p p, \quad \text{or}
\]

\[
p \leftarrow^\psi r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n,
\]

where \( x, p \in G_0, \alpha, \varphi, \psi \in G_1 \) and \( \bar{T} \in \Delta^n \). The compatibility of these two ways to write an object of \( \lvert\mathcal{G} \rtimes G_1\rvert \) can be seen by setting \( \psi = (\alpha_1 \cdots \alpha_n \varphi)^{-1} \) and recalling that groupoids ensure inverses exist (Of course, the two \( ps \) will not be equal).

Using these two descriptions for the spaces \( \lvert\mathcal{G} \rtimes G_1\rvert \), it is fairly easy to see how they are left/right \( \mathcal{G} \)-spaces. The description,

\[
r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n \leftarrow^p p,
\]

allows the definition of a right \( \mathcal{G} \)-space with anchor and action maps,

\[
\pi_R : \lvert\mathcal{G} \rtimes G_1\rvert \to G_0, \\
(r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n \leftarrow^p p) \mapsto p,
\]

\[
\mu_R : \lvert\mathcal{G} \rtimes G_1\rvert \times_{G_0} G_1 \to \lvert\mathcal{G} \rtimes G_1\rvert, \\
(r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n \leftarrow^p p, p \leftarrow^g p') \mapsto (r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n \leftarrow^{\varphi \circ g} p).
\]

Similarly, using the other description of the objects when defining \( \lvert\mathcal{G} \rtimes G_1\rvert \) as a left \( \mathcal{G} \)-space, the anchor and action maps are,

\[
\pi_L : \lvert\mathcal{G} \rtimes G_1\rvert \to G_0, \\
(p \leftarrow^\psi r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n) \mapsto p,
\]

\[
\mu_L : \mathcal{G} \rtimes G_0 \times G_1 \rvert \to \lvert\mathcal{G} \rtimes G_1\rvert, \\
p \leftarrow^g p, p \leftarrow^\psi r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n \mapsto (p' \leftarrow^{\varphi \circ g} r_0 \cdot x_0 \leftarrow^0 \cdots \leftarrow^{\alpha_n} r_n \cdot x_n).
\]

Returning to the task at hand, the elements of \( \lvert\mathcal{G} \rtimes G_1\rvert \times_{G_0} \lvert\mathcal{G} \rtimes G_1\rvert \) are
2.2. AN ANALOGUE OF MAPPING SPACES FOR ORBISPACES

of the form

$$\{(r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow x_{n+1}) \} \times G_0 \{ (y_{-1} \leftarrow \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m) \}.$$ 

The $\times G_0$ adds the requirement that the image of the anchor maps are equal, or equivalently, $x_{n+1} = y_{-1}$, these elements can now be written,

$$\{(r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow x_{n+1} \leftarrow \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m) \}.$$ 

The final step to get to the arrow space of the fibrant replacement is to take the quotient by the action of $G$ on these objects. For $g \in G_1$ with $t(g) = x_{n+1} = y_{-1}$, this quotient identifies

$$(r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow x_{n+1} \leftarrow \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m) \sim (\mu_R(r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow x_{n+1} \leftarrow \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m)),$$

$$= (r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow x_{n+1} \leftarrow \mu_L(g^{-1} \cdot y_{-1} \leftarrow \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m)),$$

$$\longrightarrow \quad \mu_R(r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow x_{n+1} \leftarrow \gamma_0^{-1} \cdot (y_{-1}) \leftarrow \gamma_0^{-1} \cdot \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m).$$

To find a standard representative for each equivalence class, note that using $g = \gamma_0$ satisfies $t(g) = y_{-1} = x_{n+1}$ which gives the representative,

$$(r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow x_{n+1} \leftarrow \gamma_0^{-1} \cdot (x_{n+1}) \leftarrow \gamma_0^{-1} \cdot (y_{-1}) \leftarrow \gamma_0^{-1} \cdot \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m).$$

Seeing as the two middle arrows are always the identity in this standard form, they are superfluous, leaving

$$(r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow x_{n+1} \leftarrow \gamma_0^{-1} \cdot (y_{-1}) \leftarrow \gamma_0^{-1} \cdot \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m).$$

Rewriting the middle element as $\beta := \alpha_{n+1} \circ \gamma_0$ gives the required result,

$$\text{mor}(fib(G)) = \{ (r_0 \cdot x_0 \leftarrow \cdots \leftarrow \alpha_n \cdot r_n \cdot x_n \leftarrow \beta \leftarrow \gamma_0^{-1} \cdot \gamma_0 \cdot y_0 \leftarrow \gamma_1 \cdots \leftarrow \gamma_m \cdot s_m \cdot y_m) \}.$$ 

Note that elements of this arrow space are subject to the same equivalence relations described above for the object space (That is, if the coefficient is 0, then drop that term and compose the surrounding arrows). The arrow $\beta$, above,
is called the middle element. Composition of arrows is defined by

\[
(r_0 \cdot x_0 \triangleleft \cdots \triangleleft \triangleleft r_n \cdot x_n \overset{\beta}{\leftarrow} s_0 \cdot y_0 \triangleleft \cdots \triangleleft s_m \cdot y_m) \\
\circ (s_0 \cdot y_0 \triangleleft \cdots \triangleleft s_m \cdot y_m \overset{\gamma_1}{\leftarrow} t_0 \cdot z_0 \triangleleft \cdots \triangleleft t_1 \cdot z_1) \\
:= (r_0 \cdot x_0 \triangleleft \cdots \triangleleft r_n \cdot x_n \overset{\beta \gamma_1}{\leftarrow} s_0 \cdot y_0 \triangleleft \cdots \triangleleft s_m \cdot y_m \overset{\gamma_1}{\leftarrow} t_0 \cdot z_0 \triangleleft \cdots \triangleleft t_1 \cdot z_1).
\]

The inverse of an arrow is defined to be

\[
(r_0 \cdot x_0 \triangleleft \cdots \triangleleft r_n \cdot x_n \overset{\beta}{\leftarrow} s_0 \cdot y_0 \triangleleft \cdots \triangleleft s_m \cdot y_m)^{-1} \\
:= (s_0 \cdot y_0 \triangleleft \cdots \triangleleft s_m \cdot y_m \overset{\alpha_1^{-1} \cdots \alpha_n^{-1} \gamma_1^{-1}}{\leftarrow} t_0 \cdot z_0 \triangleleft \cdots \triangleleft t_1 \cdot z_1) \\
:\overset{r_n \cdot x_0 \triangleleft \cdots \triangleleft r_n \cdot x_n}{\leftarrow}. \overset{r_n \cdot x_0 \triangleleft \cdots \triangleleft r_n \cdot x_n}{\leftarrow}.
\]

The identity arrow at \(r_0 \cdot x_0 \triangleleft \cdots \triangleleft r_n \cdot x_n\) is defined to be

\[
(r_0 \cdot x_0 \triangleleft \cdots \triangleleft r_n \cdot x_n \overset{\alpha_1}{\leftarrow} r_n \cdot x_n \overset{\alpha_n}{\leftarrow} r_n \cdot x_n).
\]

A functor which takes \(\mathcal{G}\) to \(\text{fib}(\mathcal{G})\) is,

\[
\mathcal{G} \rightarrow \text{fib}(\mathcal{G}) \\
\overset{x \mapsto 1 \cdot x}{\mapsto}, \text{ on objects; and} \\
(x^0 \triangleleft x^1) \mapsto (1 \cdot x^0 \triangleleft 1 \cdot x^1), \text{ on arrows.}
\]

For full details, one should refer to Section 4.2 of [GH07].

**Example 2.2.11.** Let \(M\) be a manifold. For clarity, to distinguish between \(M\) thought of as a topological groupoid and \(M\) thought of as a manifold, write \(M^*\) when \(M\) is thought of as a topological groupoid (that is, the topological groupoid with object space \(M\) and arrow space made of identity arrows). This example will describe the fibrant replacement of \(M^*\). Substituting \(M^*\) for \(\mathcal{G}\) in the definition of the fibrant replacement gives,

\[
\text{fib}(M^*) = \left( (||M^* \ltimes \text{mor}(M^*)|| \times_{\text{obj}(M^*)} ||M^* \ltimes \text{mor}(M^*)||)/M^* \Rightarrow ||M^*|| \right).
\]

The object space, \(||M^*||\) is objects of the form

\[
r_0 \cdot x_0 \triangleleft \cdots \triangleleft r_n \cdot x_n,
\]

which, recalling that the only arrows in \(M^*\) are the identity arrows, becomes

\[
r_0 \cdot x_0 \overset{id}{\leftarrow} \cdots \overset{id}{\leftarrow} r_n \cdot x_0,
\]

where \(x_0 \in M\) and \(\vec{r} \in \Delta^n\). For each point in \(x_0 \in M\), there is a corresponding
2.2. AN ANALOGUE OF MAPPING SPACES FOR ORBISPACES

space in \(||M^*|||\), lying over \(x_0\), which is homeomorphic to

\[
\bigsqcup_{n \geq 0} \Delta^n / \sim
\]

where the equivalence relation is given by identifying the \(n\)-simplex with each face of the \(n+1\)-simplex. Note that the space \(\bigsqcup_n \Delta^n / \sim\) coincides with the definition of the fat geometric realisation \(||[*/1]|||\) where 1 is the group on one element. This gives

\[
| |M^*|| = M \times | |[*/1]||.
\]

which can be thought of as a “fatter” version of \(M\).

The arrow space of \(\text{fib}(M^*)\) is objects of the form

\[
r_0 \cdot x \leftarrow \cdots \leftarrow r_n \cdot x \leftarrow s_0 \cdot x \leftarrow \cdots \leftarrow s_m \cdot x,
\]

for \(x \in M, r \in \Delta^n\) and \(s \in \Delta^m\). That is, if \(x, y \in M, r \in \Delta^n\) and \(s \in \Delta^m\),

1. if \(x = y\) then there is a unique arrow,

\[
(r_0 \cdot x \leftarrow \cdots \leftarrow r_n \cdot x) \leftarrow (s_0 \cdot y \leftarrow \cdots \leftarrow s_m \cdot y),
\]

coming from the identity arrow at \(x = y\); and

2. if \(x \neq y\) then there exists no such an arrow.

Note here that taking the coarse space of the fibrant replacement for \(M\) (that is, the fibrant replacement of the corresponding topological groupoid \(M^*\)) indeed results in getting \(M\) back again. This shows that the act of taking the fibrant replacement of \(M\) has not done anything too drastic to its structure.

**Example 2.2.12.** Let \(H\) be a topological group. This example will look into the fibrant replacement of the groupoid \([*/H]\). Substituting for \(G\) in the definition of fibrant replacement gives,

\[
\text{fib}([*/H]) = ((||[*/H] \times H|| \times_{*} ||[*/H] \times H||)/ [*/H] \Rightarrow ||[*/H]||).
\]

The objects, \(| |[*/H]|||\), are elements

\[
(r_0 \cdot * \leftarrow \cdots \leftarrow r_n \cdot *),
\]

where \(\alpha_1, \ldots, \alpha_n \in H\) and \(\bar{r} \in \Delta^n\) which are identified using a slightly more complicated equivalence condition than in the \(\text{fib}(M)\) case. Since it is possible
to multiply any combination $\alpha_1, \ldots, \alpha_n \in H$, the object space is

$$\text{obj}(\text{fib}([*/H])) = \bigsqcup_{n \geq 0} H^n \times \Delta^n / \sim,$$

where $\sim$ is the equivalence relation generated by,

1. $(h_1, h_2, \ldots, h_n, 0, t_1, \ldots, t_n) \sim (h_2, \ldots, h_n, t_1, \ldots, t_n)$;
2. $(h_1, \ldots, h_n, t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \sim (h_1, \ldots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \ldots, h_n, t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$; and
3. $(h_1, \ldots, h_{n-1}, h_n, t_0, \ldots, t_{n-1}, 0) \sim (h_1, \ldots, h_{n-1}, t_0, \ldots, t_{n-1})$.

Intuitively, if a point in a simplex is moved onto the boundary of $\Delta^n$, then the corresponding group elements can be multiplied together. In the current chapter, this description of the object space will be written $BH$, the fat classifying space of $H$. Note that the usual definition of $BH$ includes an extra equivalence relation, however these two descriptions result in spaces which are homotopic to one-another (This other definition appears in a later chapter, Definition 4.1.3). As an aside, the fibrant replacement above could be written instead:

$$EH \times_H EH \cong EH/H,$$

where $EH$ is the universal cover of $BH$.

Returning to the example, the arrows between any two objects are of the form

$$(r_0 \cdot \epsilon^\alpha_1 \cdot \ldots \cdot \epsilon^\alpha_n \cdot r_n \cdot \epsilon^\beta \cdot s_0 \cdot \epsilon^\gamma_1 \cdot \ldots \cdot \epsilon^\gamma_m \cdot s_m \cdot *)$$

where $\beta$ is any element of $H$. So,

$$\text{Hom}((\vec{a}, r), (\vec{\gamma}, s)) = H.$$ 

The result is that $\text{fib}([*/H])$ is the topological groupoid with

- object space $BH$, the fat geometric realisation of $H$; and
- arrows between $x, y \in BH$ being $\text{Hom}(x, y) = H$.

Having unpacked the fibrant replacement of select groupoids, consider the description of the mapping spaces on orbifolds.

**Proposition 2.2.13.** Let $M$ be a manifold, $\mathcal{H}$ be a topological groupoid and let $H$ be a topological group. Then,

1. $\text{map}(M, \text{fib}(\mathcal{H})) = \text{map}(M, |\mathcal{H}|)$; and
2.2. AN ANALOGUE OF MAPPING SPACES FOR ORBISPACES

\[ \text{map}(M, \text{fib}(\ast/H)) = \text{map}(M, BH), \]

where \( B \) denotes the fat geometric realisation. The mapping spaces on the left-hand-side are the spaces of continuous functors between the two orbispaces, the mapping spaces on the right-hand-side are the spaces of continuous maps between topological spaces.

\[ \text{Proof.} \text{ Let } \mathcal{H} \text{ be a topological groupoid and let } M \text{ be a manifold. As before, in this proof write } M^* \text{ for the topological groupoid with object space being the manifold } M \text{ and arrow space being precisely the identity arrows. This proof will investigate the mapping space } \text{map}(M^*, \text{fib}(\mathcal{H})). \]

This problem can be generalised by looking at the mapping space \( \text{map}(M^*, \mathcal{G}) \) for some topological groupoid \( \mathcal{G} \). Suppose that \( f \) is an element of \( \text{map}(M^*, \mathcal{G}) \). Then \( f \) is a continuous functor, which is a continuous map \( f_0 : \text{obj}(M^*) \to \text{obj}(\mathcal{G}) \) on objects and a continuous map \( f_1 : \text{mor}(M^*) \to \text{mor}(\mathcal{G}) \) on arrows. At this point, it is important to note that since the only arrows in \( M^* \) are the identity arrows, the only arrows in the image of \( f_1 \) will also be identity arrows. Therefore, the continuous map on objects, \( f_0 : \text{obj}(M^*) \to \text{obj}(\mathcal{G}) \), completely defines the functor \( f \). The result is that for \( M \) a manifold,

\[ \text{map}(M^*, \mathcal{G}) = \text{map}(M, \text{obj}(\mathcal{G})), \]

where the space on the left is a mapping space between topological groupoids and the space on the right is a mapping space between topological spaces.

This is immediately applicable to the mapping space \( \text{map}(M^*, \text{fib}(\mathcal{H})) \). Indeed, the result above simply gives

\[ \text{map}(M^*, \text{fib}(\mathcal{H})) = \text{map}(M, \text{obj}(\text{fib}(\mathcal{H}))), \]

\[ = \text{map}(M, ||\mathcal{H}||), \]

where the left is maps of topological groupoids and the right is maps of topological spaces. Furthermore, in the special case when \( \mathcal{H} = \ast/H \), where \( H \) is a topological group, the results above give

\[ \text{map}(M^*, \text{fib}(\mathcal{H})) = \text{map}(M, BH). \]

As a note, the space of maps \( \text{map}(M, BH) \) classify equivalence classes of principal \( H \)-bundles over \( M \), see [Str11], for example.

\[ \square \]

Returning to the functor \( R \) taking topological groupoids to Orb-spaces, recall
the definition for a topological groupoid $G$,

$$R(G) : \mathcal{F} \to \{\text{Spaces}\},$$

$$H \mapsto \text{map}(\text{fib}(\ast/H), \text{fib}(G)).$$

In general, these mappings could be quite complicated. There are, however, some cases which can be simplified. One of these simple cases is the space $\text{map}(\text{fib}(\ast/H), \text{fib}(M))$ where $H$ is a topological group and $M$ is a manifold (or, the topological groupoid with object space $M$ and arrow space identity arrows).

**Proposition 2.2.14.** Let $M$ be a manifold, and let $H$ be a topological group. Then

$$\text{map}(\text{fib}(\ast/H), \text{fib}(M)) = M \times \text{map}(BH, ||\ast/1||),$$

where $B$ denotes the fat geometric realisation. As with the previous proposition, the mapping spaces on the left-hand-side are the spaces of continuous functors between the two orbispaces and the mapping spaces on the right-hand-side are the spaces of continuous maps between topological spaces.

**Proof.** Consider the space $\text{map}(\text{fib}(\ast/H), \text{fib}(M^*))$. Recall from Example 2.2.11 that $\text{fib}(M^*)$ is a ‘fattened up’ version of $M$ along with arrows, and that from Example 2.2.12 that $\text{fib}(\ast/H)$ is the orbispace with

- object space $BH$ (the fat classifying space of $H$); and
- arrow space $\text{Hom}(x_1, x_2) = H$, where $x_1$ and $x_2$ are any points in $BH$.

The mapping space is the space of continuous functors, $f$. On objects, such a functor is a continuous map:

$$f_0 : BH \to M \times ||\ast/1||.$$

A point $x_1 \in BH$ maps to $f_0(x_1) = (m^1, t^1) \in M \times ||\ast/1||$. However, this map on objects needs to commute with the map on arrows,

$$f_1 : (x_1 \to x_2) \mapsto ((m^1, t^1) \to (m^2, t^2)),$$

where $(m^i, t^i) = f_0(x_i)$. Recall that in the right-hand-side ($\text{fib}(M^*)$), there is a unique arrow between $(m^1, t^1)$ and $(m^2, t^2)$ if and only if $m^1 = m^2$ and such an arrow comes from the identity arrow in $\ast/1$. Furthermore, recall that on the left-hand-side ($\text{fib}(\ast/H)$), there is always an arrow between any two $x_1$ and
2.3. THE CONFIGURATION ORBISPACE

Let \( x_2 \) in \( BH \). This implies that, in order for \( f \) to be a functor, the image of \( BH \) must lie over a single point in \( M \).

Therefore, \( f_0 \) is a continuous map of the form

\[ BH \to \{m\} \times \|\{1\}\|, \]

for some \( m \in M \). Indeed, taking \( f_0 \) to be any such map defines a continuous functor. Consider any two elements, \( x_1, x_2 \in BH \). The space of arrows \( x_1 \to x_2 \) is \( H \). In \( \|\{1\}\| \), the only arrow \( f_0(x_1) \to f_0(x_2) \) comes from the identity arrow in \( \{1\} \). So every arrow coming from an element of \( H \) on the left-hand-side, must map to the arrow given by the identity arrow in \( \|\{1\}\| \) under \( f_1 \). The functor given by \( f_0 \) and \( f_1 \) preserves identity arrows and composition.

The mapping space is therefore

\[ \text{map}(\text{fib}(\{1\}), \text{fib}(M^*)) = M \times \text{map}(BH, \|\{1\}\|). \]

Compare this to the mapping space \( \text{map}(\{1\}, M^*) \), an element of which maps the point \( * \) to a point \( m \in M \) and maps all of the group elements \( H \) to the identity morphism at \( m \), giving the result

\[ \text{map}(\{1\}, M^*) = M. \]

\[ \square \]

2.3 The Configuration Orbispace

In this section the concept of the configuration orbispace is introduced, along with the idea of relative configuration orbispace and ghost points.

Recall the definition of the configuration space of a topological space \( X \).

**Definition 2.3.1.** Let \( X \) be a topological space and \( n \in \mathbb{Z}_{\geq 0} \). The **ordered configuration space** or **pure configuration space** of \( n \) points in \( X \) is the space of ordered, distinct points in \( X \):

\[ P\text{Conf}_n(X) := \{ \vec{x} \in X^n \mid \text{if } i \neq j \text{ then } x_i \neq x_j \}. \]

The **unordered configuration space** of \( n \) points in \( X \) is obtained from the ordered configuration space by taking the quotient by \( S_n \), the symmetric group on \( n \) objects,

\[ \text{Conf}_n(X) := P\text{Conf}_n(X)/S_n. \]
A long held belief was that the homotopy type of the configuration space was only an invariant on the dimension and homotopy type of the underlying manifold. This was proved not to be the case by Longoni and Salvatore (see \cite{LS05}). This subtle result has parallels with the homology of configurations on orbifolds. Indeed, the counterexample of Longoni and Salvatore used configuration spaces of lens spaces, quotients of $S^3$ by $\mathbb{Z}_p$. In their paper, these lens spaces were treated as manifolds, but they can just as easily be thought of as orbifolds.

There are a few changes that should be made to this definition to get the configuration orbispace of an orbispace $X$. Take two points in the object space $x_1, x_2 \in X_0$ such that there is an arrow $x_1 \rightarrow x_2$, then it is advantageous to think of $x_1$ and $x_2$ being almost the same point (taking the coarse space of $X$, $x_1$ and $x_2$ actually become the same point). This suggests that when defining the configuration ‘space’ of an orbispace, rather than just requiring that the points be distinct, the requirement should be expanded to say that the points should have distinct orbits. In defining the arrow space of the configuration orbispace, an arrow should allow any point to move along an arrow in $X_1$. So an arrow of the configuration orbispace will be made up of a tuple of arrows from $X_1$.

The reasoning above leads to the idea of a ghost point which is thought of as a point that, even though there isn’t actually anything there, the other points in the configuration are restricted from passing over it\footnote{So quite dissimilar to a real ghost.}. Alternatively, for a point $x \in X_0$, $x$ could be thought of as consisting of its entire orbit under arrows in $X_1$, even though only one of these points is actually chosen.

**Definition 2.3.2.** Let $X$ be an orbispace and let $x \in X_0$. A ghost point of $x$ is an element of the set

$$\{ y \in X_0 \mid y \neq x \text{ and there is an arrow } y \rightarrow x \}. $$

Figure 2.4 shows an example of a point in $[D/\mathbb{Z}_2]$ with its two ghost points. Note that, in this case, if $x$ was at the origin, then it would have no ghost points.

The definition of the configuration orbispace can now be constructed so that any point $x_i$ cannot equal $x_j$ or equal any of $x_j$’s ghost points.

**Definition 2.3.3.** Let $X$ be an orbispace and $n \in \mathbb{Z}_{\geq 0}$. Then the ordered configuration orbispace or pure configuration orbispace of $n$ points in $X$ is the orbispace, written $PConf_n(X)$, with

- object space

$$\{ \tilde{x} \in X_0^n \mid \text{if } i \neq j \text{ then there is no arrow } x_i \rightarrow x_j \};$$
and

* morphism space being \( n \)-tuples of morphisms from \( X_1 \). That is, there is an arrow

\[
(x_1, \ldots, x_n) \to (y_1, \ldots, y_n),
\]

for every choice of \( n \) arrows in \( X_1 \) such that \( x_1 \to y_1, \ldots, x_n \to y_n \).

Alternatively, one can define the ordered configuration orbispace as simply the full subcategory of \( X^\times_n \) on \( (x_1, \ldots, x_n) \) such that if \( i \neq j \) then there does not exist an arrow \( x_i \to x_j \) in \( X_1 \). Note the similarities to the definition of the configuration space of a topological space \( X \), where \( PConf_n(X) \) is the subspace of \( X^\times_n \) on \( (x_1, \ldots, x_n) \) such that if \( i \neq j \) then \( x_i \neq x_j \).

In the case of \( X = [M/G] \), one can note the similarities of this definition with that of the orbit configuration space,

\[
F_G(M,k) := \{(m_1, \ldots, m_k) \in M^k | \text{if } i \neq j \text{ then } Gm_i \neq Gm_j \}.
\]

For an investigation of the orbit configuration space, see [XAM97].

As before, the unordered configuration orbispace is defined by quotiening out by the symmetric group.

**Definition 2.3.4.** Let \( \mathcal{X} \) be an orbispace and \( n \in \mathbb{Z}_{\geq 0} \). The *unordered configuration orbispace* of \( n \) points in \( \mathcal{X} \) is the orbispace \( Conf_n(\mathcal{X}) \) with

* object space \( \text{obj}(Conf_n(\mathcal{X})) = \text{obj}(PConf_n(\mathcal{X}))/S_n \); and

* morphism space \( \text{mor}(Conf_n(\mathcal{X})) = \text{mor}(PConf_n(\mathcal{X}))/S_n \).

The reader may be concerned by what is meant when saying that the morphism space is \( \text{mor}(PConf_n(\mathcal{X}))/S_n \), and whether it is well defined. Take two
sets of points \( x, y \in \text{obj}(Conf_n(X))/S_n \). Let \( \vec{x}^1, \ldots, \vec{x}^{n!} \) be the \( n! \) ordered configurations in \( \text{obj}(PConf_n(X)) \) which lie over \( x \), similarly for \( \vec{y}^1, \ldots, \vec{y}^{n!} \) over \( y \).

**Claim 2.3.5.** There exists at most one \( \rho \in S_n! \) such that all of

\[
B^\rho_i := \{ \vec{\beta} \in \text{mor}(PConf_n(X)) \mid \vec{\beta} : \vec{x}^i \rightarrow \vec{y}^{\rho(i)} \}
\]

are non-empty. Furthermore, if there is a \( \rho^* \in S_n! \) and an \( i \in \{1, \ldots, n!\} \) such that \( B^\rho_i \neq \emptyset \), such a \( \rho \) does exist. Then, given \( i, j \in \{1, \ldots, n!\} \), there exists \( \sigma \in S_n \) such that,

\[
\begin{align*}
B^\rho_i & \xrightarrow{\cong} B^\rho_j, \\
(\vec{\beta} : \vec{x}^i \rightarrow \vec{y}^{\rho(i)}) & \mapsto (\sigma(\vec{\beta}) : \sigma(\vec{x}^i) \rightarrow \sigma(\vec{y}^{\rho(i)})).
\end{align*}
\]

Once this claim has been shown, the arrows from \( x \) to \( y \) in \( Conf_nX \) are defined to be,

1. \( \text{Hom}(x, y) := \emptyset \), if \( \rho \) as described in the claim does not exist; or
2. \( \text{Hom}(x, y) := (\cup_i B^\rho_i)/S_n \), if such a \( \rho \) does exist.

**Proof of Claim.** Beginning with the ‘at most one \( \rho \)' condition, suppose that there were two distinct elements \( \rho, \rho^* \in S_n! \) such that every \( B^\rho_i \) and every \( B^\rho_* \) were non-empty. Since \( \rho \neq \rho^* \), there exists \( k \in \{1, \ldots, n!\} \) such that \( \rho(k) \neq \rho^*(k) \). Therefore, there exist two arrows,

\[
\begin{align*}
(\vec{\beta} : \vec{x}^k \rightarrow \vec{y}^{\rho(k)}) & \in B^\rho_k, \\
(\vec{\beta}^* : \vec{x}^k \rightarrow \vec{y}^{\rho^*(k)}) & \in B^{\rho^*}_k.
\end{align*}
\]

Using these arrows gives the composition,

\[
\vec{\beta}^* \circ \vec{\beta}^{-1} : \vec{y}^{\rho(k)} \rightarrow \vec{y}^{\rho^*(k)}.
\]

Since \( \rho(k) \neq \rho^*(k) \), and the \( \vec{y} \) were the \( n! \) different orderings of \( y \), there must exist \( l \in \{1, \ldots, n\} \) such that \( y_l^{\rho(k)} \neq y_l^{\rho^*(k)} \). However, there is an arrow between \( \vec{y}^{\rho(k)} \) and \( \vec{y}^{\rho^*(k)} \), which implies there is an arrow \( y_l^{\rho(k)} \rightarrow y_l^{\rho^*(k)} \). This is a contradiction on the assumption that each of the \( \vec{y} \) were elements of \( \text{obj}(PConf_nX) \), as well as \( y \) an element of \( \text{obj}(Conf_nX) \), as two points are not allowed to be connected by an arrow. Therefore there can be at most one \( \rho \in S_n! \) such that all of the \( B^\rho_i \) are non-empty.

For the remainder of the claim, assume that there is \( \rho^* \in S_n! \) and \( k \in \{1, \ldots, n!\} \) such that \( B^{\rho^*}_k \) is non-empty. That means there exists an arrow
2.3. THE CONFIGURATION ORBISPACE

33

\[ \vec{y} : \vec{x}^k \to \vec{y}^{(k)} \]. The aim is to construct an element \( \rho \in S_{n!} \) so that \( B_i^\rho \) is non-empty for every \( i \). Let \( i \in \{1, \ldots, n!\} \), then there exists \( \sigma_i \in S_n \) such that \( \vec{x}^i = \sigma_i(\vec{x}^k) \), since \( \vec{x}^i \) and \( \vec{x}^k \) are both orderings of the elements of \( \vec{x} \). Since arrows in \( PConf_n \mathcal{X} \) are made up of \( n \) arrows from \( \mathcal{X} \), there is an arrow,

\[
\begin{align*}
\sigma_i(\vec{b}) : \sigma_i(\vec{x}^k) &\to \sigma_i(\vec{y}^{(k)}), \\
\sigma_i(\vec{b}) : \vec{x}^i &\to \sigma_i(\vec{y}^{(k)}).
\end{align*}
\]

Define \( \rho \) to be the element of \( S_{n!} \) so that if \( i \in \{1, \ldots, n!\} \) then,

\[ \vec{y}^{(i)} = \sigma_i(\vec{y}^{(k)}) \).

Such an element exists since the \( \vec{y}^i \) are all possible orderings of the elements of \( y \). Then \( (\sigma_i(\vec{b}) : \vec{x}^i \to \vec{y}^{(i)}) \in B_i^\rho \). With this \( \rho \), every \( B_i^\rho \) is non-empty as required.

Furthermore, repeating this argument for every element of \( B_k^{\rho^*} \) will yield:

\[
B_i^\rho = \{ \sigma_i(\vec{b}) : \vec{x}^i \to \vec{y}^{(i)} \mid (\vec{b} : \vec{x}^k \to \vec{y}^{(k)}) \in B_k^{\rho^*} \}.
\]

The isomorphism at the end of the claim is obtained by,

\[
B_i^\rho \overset{\cong}{\to} B_j^\rho,
\]

\[
(\vec{b} : \vec{x}^i \to \vec{y}^{(i)}) \mapsto (\sigma(\vec{b}) : \sigma(\vec{x}^i) \to \sigma(\vec{y}^{(i)})),
\]

with \( \sigma = \sigma_j \circ \sigma_i^{-1} \).

**Proposition 2.3.6.** In the case that \( \mathcal{X} \) is an orbifold, an alternate definition for the unordered configuration orbispace via the ordered configuration orbispace is to take

- the object space of \( Conf_n(\mathcal{X}) \) equal to the object space of \( PConf_n(\mathcal{X}) \); and

- the morphism space of \( Conf_n(\mathcal{X}) \) to be \( \text{mor}(PConf_n(\mathcal{X})) \times S_n \),

so that a reordering of the points in a configuration corresponds to an arrow.

In this case there are more objects, but anything that would have been identified by the quotient in the first definition, are now connected by an arrow in the second definition. A result of this simple argument is that taking the coarse space from either of these definitions leads to the same topological space, though showing that these are equivalent orbispaces is somewhat more involved.

**Proof.** Let \( \mathcal{X} \) be an orbifold. To show that this alternate definition is equivalent, one must find a Morita equivalence between the two definitions of unordered
configuration orbispaces. Let \( \mathcal{G} \) and \( \mathcal{H} \) be defined by the two different ways of defining the unordered configuration orbispace. Then,

\[
\begin{align*}
G_0 &= \text{obj}(P\text{Conf}_n \mathcal{X})/S_n, & H_0 &= \text{obj}(P\text{Conf}_n \mathcal{X}), \\
G_1 &= \text{mor}(P\text{Conf}_n \mathcal{X})/S_n, & H_1 &= \text{mor}(P\text{Conf}_n \mathcal{X}) \times S_n.
\end{align*}
\]

There is an obvious map \( \varphi : \mathcal{H} \to \mathcal{G} \) given by,

\[
\varphi : \mathcal{H} \to \mathcal{G},
\]

\[
(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\},
\]

\[
((\alpha_1, \ldots, \alpha_n), \sigma) \mapsto \{\alpha_1, \ldots, \alpha_n\}.
\]

Recalling the definition of Morita equivalence (Definition 2.1.14), there are two conditions that need to be checked:

1. the map

\[
t \circ \text{proj}_1 : G_1 \times \varphi H_0 \to G_0,
\]

\[
(g, y) \mapsto t(g)
\]

is a surjective local homeomorphism; and

2. the square

\[
\begin{array}{ccc}
H_1 & \xrightarrow{\varphi} & G_1 \\
\downarrow_{(s,t)} & & \downarrow_{(s,t)} \\
H_0 \times H_0 & \xrightarrow{\varphi \times \varphi} & G_0 \times G_0
\end{array}
\]

is a fibred product.

For the first condition, the map is defined on the space

\[
\{(g, y) \mid g \in \text{mor}(P\text{Conf}_n \mathcal{X})/S_n, y \in \text{obj}(P\text{Conf}_n \mathcal{X}), s(g) = \varphi(y)\}.
\]

The map \( t \circ \text{proj}_1 \) is indeed surjective. A point

\[
\{y_1, \ldots, y_n\} \in G_0 = \text{obj}(P\text{Conf}_n \mathcal{X})/S_n,
\]

in the co-domain is the image of the point,

\[
(id_{\{y_1, \ldots, y_n\}}, (y_1, \ldots, y_n)) \in G_1 \times \varphi H_0.
\]

Claim that the map \( \varphi_0 : H_0 \to G_0 \) is a local homeomorphism. For the
configuration $y = \{y_1, \ldots, y_n\} \in G_0$, there is a neighbourhood around each point $U_1, \ldots, U_n \subseteq X_0$ such that if $i \neq j$ then $U_i \cap U_j = \emptyset$ (by the Hausdorff assumption on orbifolds and $y_i \neq y_j$). For a point in the pre-image of $y$ under $\varphi$, say $(y_{\sigma(1)}, \ldots, y_{\sigma(n)})$ for $\sigma \in S_n$, these $U_i$ define an open neighbourhood in $H_0$. The open neighbourhood is $U_{\sigma(1)} \times \cdots \times U_{\sigma(n)} \subseteq H_0$ which, since the $U_i$ are disjoint, is homeomorphic to the open neighbourhood $(\cup_i U_i)^{\times n}/S_n \subseteq G_0$.

To see that the map in the first condition is a local homeomorphism, take a point $y = \{y_1, \ldots, y_n\} \in G_0$. Take a point, $(\alpha, \bar{x}) \in G_{1, x} \times \varphi H_0$, in the pre-image of $y$. Then $\bar{x}$ is a tuple of points $(x_1, \ldots, x_n) \in \text{obj}(PConf_n X)$ such that the arrows $\alpha_i$ satisfy $\alpha_i : x_{\sigma(i)} \to y_i$ for some $\sigma \in S_n$. Since the target map is a local homeomorphism (by the definition of orbifold), a neighbourhood of $y$ is homeomorphic to a neighbourhood of $\bar{x}$, say $\bar{x} \in \bar{U} \subseteq G_1$. Furthermore, since the source map is a local homeomorphism, and $\varphi$ is a local homeomorphism, there is some neighbourhood of $\bar{x}$, say $\bar{x} \in V \subseteq G_1$, which is homeomorphic to a neighbourhood of $(\alpha, \bar{x})$. It is then the case that the intersection of both of these neighbourhoods, $U \cap V$, will be homeomorphic to both a neighbourhood of $(\alpha, \bar{x})$ and a neighbourhood of $y$. Therefore $t \circ \text{proj}_1$ is a local homeomorphism and the first condition for $\varphi$ to be a Morita equivalence is satisfied.

The second condition is to show that the square above is a fibred product. That is, if there is another space $U$ with maps $q_1 : U \to H_0 \times H_0$ and $q_2 : U \to G_1$ such that

$$
\begin{array}{ccc}
U & \xrightarrow{q_2} & G_1 \\
q_1 \downarrow & & \downarrow (s,t) \\
H_0 \times H_0 & \xrightarrow{\varphi \times \varphi} & G_0 \times G_0,
\end{array}
$$

commutes, then there is a map $u : U \to H_1$ such that $(s, t) \circ u = q_2$ and $\varphi \circ u = q_1$. By the commutative diagram, if $x \in U$ then $(s, t) \circ q_2(x) = (\varphi \times \varphi) \circ q_1(x)$. The image of $x$ in $G_0 \times G_0$ is $(y^1, y^2)$ such that there is an arrow in $G_1$, $q_2(u) = g : y^1 \to y^2$. The element $(y^1, y^2)$ also has to map to an element $q_1(x) = (y^3, y^4)$ in $H_0 \times H_0$, where each $y^i$ is an ordering of $y^1$. Note that $y^3$ and $y^4$ may not necessarily have the ‘same’ ordering. There is, however, still an arrow $h : y^3 \to y^4$ in $H_1$. Indeed, this is the case by the following two observations:

1. $H_1$ consists of arrows on each co-ordinate, followed by a re-ordering of the co-ordinates; and

2. $q_2(u)$ is an arrow between the unordered points $y^1$ and $y^2$.

Repeating this process for every $x \in U$, then setting $u(x) := h$ will show that
the diagram is a fibred product.

The map $\varphi$ is therefore a Morita equivalence, and the two orbispaces are Morita equivalent.

**Proposition 2.3.7.** Let $\mathcal{X}$ be an orbispace. Then,

$$|PConf_n(\mathcal{X})| \cong PConf_n(|\mathcal{X}|), \quad \text{and}$$

$$|Conf_n(\mathcal{X})| \cong Conf_n(|\mathcal{X}|),$$

where the left-hand-side is the coarse space of the configuration orbispace and the right-hand-side is the classical configuration space on topological spaces.\(^3\)

**Proof.** The proof of the first equation will be presented here, the second equation can be shown in a similar fashion. To convince the reader of this identity, a bijection will be found between the points of $|PConf_n\mathcal{X}|$ and the points of $PConf_n\mathcal{X}$.

Consider the orbispace $\mathcal{X}$ with object space $X_0$ and arrow space $X_1$. The coarse space $|PConf_n\mathcal{X}|$ is made up of equivalence classes $[(x_1, \ldots, x_n)]$, where $(x_1, \ldots, x_n) \in \text{obj}(PConf_n\mathcal{X})$. Similarly, points in $PConf_n\mathcal{X}$ are of the form $([x_1], \ldots, [x_n])$, where the equivalence classes, $[x_i]$, are distinct and $x_i \in \text{obj}(\mathcal{X})$.

The map,

$$\varphi : |PConf_n\mathcal{X}| \to PConf_n\mathcal{X}, \quad [(x_1, \ldots, x_n)] \mapsto ([x_1], \ldots, [x_n]),$$

is a bijection, with inverse,

$$\psi : PConf_n\mathcal{X} \to |PConf_n\mathcal{X}|, \quad ([x_1], \ldots, [x_n]) \mapsto [(x_1, \ldots, x_n)].$$

At first sight, the way this is written makes the fact that this is a bijection seem obvious. However, the reader should become wary upon the realisation that this bijection is messing with the danger that is equivalence classes. Indeed, each equivalence class could be labelled by any element of that class. What needs to be checked, then, is that distinct $[(x_1, \ldots, x_n)]$ and $[(y_1, \ldots, y_n)]$ produce distinct images under $\varphi$, and vice-versa.

To achieve the goal, pick a unique set of labels for the equivalence classes. That is, a set $S \subseteq X_0$ such that

- if $x, y \in S$ and $x \neq y$, then $[x] \neq [y]$; and

\(^3\)The reader is encouraged to refer to the coarse space functor when applied to a configuration orbispace as the ‘ghostbuster’ as it eliminates all the ghost points. The author, however, will refrain from using this terminology.
• if \( z \in X_0 \) then there exists \( x \in S \) such that \( [x] = [z] \).

The claim is that every element equivalence class in \( |PConf_nX| \) can be written uniquely as \( ([x_1, \ldots, x_n]) \) with \( x_i \in S \), and similarly for elements of \( PConf_n|X| \). Begin by looking at the case of \( |PConf_nX| \), the two things that are needed to be shown are that every equivalence class can be represented using elements of \( S \) and that this representation is unique. Take an equivalence class \( ([y_1, \ldots, y_n]) \), where \((y_1, \ldots, y_n) \in \text{obj}(PConf_n|X|)\). Recall that the space of arrows in \( PConf_n|X| \) is made up of an \( n \)-tuple of arrows in \( X_1 \). From each \( y_i \) there is an arrow to some \( x_i \in S \), by the second condition in the definition of \( S \), say \( \alpha_i : y_i \rightarrow x_i \). These \( \alpha_i \) give an arrow in \( PConf_n|X| \), namely:

\[
(\alpha_1, \ldots, \alpha_n) : (y_1, \ldots, y_n) \rightarrow (x_1, \ldots, x_n).
\]

This implies that upon taking the coarse space of \( PConf_n|X| \) that

\[
[(y_1, \ldots, y_n)] = [(x_1, \ldots, x_n)].
\]

The uniqueness of this choice of \( (x_1, \ldots, x_n) \) is immediate from the first condition in the definition of \( S \). As if there was an \( x'_i \in S \) such that \( y_i \rightarrow x'_i \) with \( x_i \neq x'_i \) then it would follow that \( [x_i] = [x'_i] \), a contradiction.

It is much simpler to show the uniqueness of representations on the configuration space \( PConf_n|X| \). Begin by taking \( ([y_1], \ldots, [y_n]) \in PConf_n|X| \). Then the definition of \( S \) implies that for each \( i \), there is a unique choice of \( x_i \in S \) such that \( [x_i] = [y_i] \). The tuple \( ([x_1], \ldots, [x_n]) \) is the unique representative such that,

\[
([x_1], \ldots, [x_n]) = ([y_1], \ldots, [y_n]).
\]

This shows that \( \varphi \) is a bijection from \( |PConf_nX| \) to \( PConf_n|X| \) with inverse \( \psi \).

In later sections when referring to the homology or homotopy of an orbispace or configuration orbispace, the following definition will be of use.

**Definition 2.3.8.** Let \( \mathcal{X} \) be an orbispace and let \( R \) be a commutative ring. Define,

\[
H_*(\mathcal{X}; R) := H_*(BX; R),
\]

\[
H^*(\mathcal{X}; R) := H^*(BX; R),
\]
the homology and cohomology of $X$ with co-efficients in $R$, and

$$\pi_*(X) := \pi_*(BX),$$

the homotopy groups of $X$, where $BX$ represents the classifying space of $X$.

The choice of using the classifying space to define homotopy and homology groups is common in category theory. For example, taking the cohomology of $[\ast/G]$ gives,

$$H^*(B[\ast/G]; \mathbb{Z}) = H^*(G; \mathbb{Z}),$$

the group cohomology of $G$. Similarly for the homotopy groups of $[\ast/G],$

$$\pi_1(B[\ast/G]) = G,$$
$$\pi_{\geq 2}(B[\ast/G]) = 0,$$

as expected for the homotopy of the group $G$.

Also worth noting is that using the classifying space to obtain the fundamental group of an orbifold coincides with the classical definition via the natural extension of the embeddings of $S^1$. As a reminder, this extension brings the idea of loops to orbifolds. Thinking of an orbifold of the form $[M/G]$, a loop is allowed to ‘jump’ along group elements in $G$ (the loop now appears to be discontinuous in $M$). Taking the equivalence class of these paths under homotopy, followed by quotienting by $G$, gives the fundamental group.

Finally, the definition of the homotopy fibre and quasi-fibration follow.

**Definition 2.3.9.** Let $f : X \to Y$ be a continuous map between topological spaces. The *homotopy fibre* over a fixed base-point $y \in Y$ is the space

$$\text{hofib}_y(f) := \{ (x, \gamma) \mid x \in X, \gamma : I \to Y, \gamma(0) = f(x), \gamma(1) = y \}.$$

**Definition 2.3.10.** Let $f : X \to Y$ be a continuous map between topological spaces. Then $f$ is called a *quasi-fibration* if whenever $y \in Y$ then the canonical map between the fibre over $y$ and the homotopy fibre over $y$,

$$f^{-1}(y) \to \text{hofib}_y(f),$$
$$z \mapsto (z, c_y),$$

is a homotopy equivalence, where $c_y$ is the constant path at $y$.

Both of these definition will play an important role in Chapter 4.
2.4 Examples Relating to the Functor from Topological Groupoids to $Orb$-spaces

This section will be concerned with investigating the values taken by $R(\mathcal{G})$ for different groups. The simplest case is calculating the space $R(\mathcal{G})(1)$, where $\mathcal{G}$ is any topological groupoid and 1 is the trivial group with one element. Recall that

$$R(\mathcal{G})(H) := \text{map}(\text{fib}([*/H]), \text{fib}(\mathcal{G})), $$

where $H$ is a group and the mapping space is made up of continuous functors from $\text{fib}([*/H])$ to $\text{fib}(\mathcal{G})$. When $H = 1$, $\text{fib}([*/1])$ is the topological groupoid with object space $||[*/1]||$, the fat geometric realisation, an infinite dimensional contractible space, and exactly one arrow between every two objects of $||[*/1]||$.

On objects, the map is a continuous map of the form

$$||[*/1]|| \rightarrow \text{obj}(\text{fib}(\mathcal{G})), $$

$$= ||\mathcal{G}||.$$

As any two points on the left-hand side are connected by an arrow, any two points in the image of the above map must be connected by an arrow in $\text{fib}(\mathcal{G})$. This is not limited to the case when the co-domain is a fibrant replacement. Indeed, if $\mathcal{G}$ is any topological groupoid, the space of functors

$$\text{map}(\text{fib}([*/1]), \mathcal{G})$$

consists of continuous maps

$$||[*/1]|| \rightarrow \text{obj}(\mathcal{G})$$

such that any two objects in the image are connected by an arrow.

Consider the example where $\mathcal{G} = [D^2/\mathbb{T}]$. The map on objects is a continuous map

$$f_0 : ||[*/1]|| \rightarrow D^2$$

so that any two points in the image are connected by an arrow. Pick any $x \in ||[*/1]||$, then there are two possibilities for $f_0(x)$, either

1. $f_0(x) = 0$, the origin in $D^2$; or

2. $f_0(x) \neq 0$. 

Investigating the first case, recall the ‘every two points in the image must be connected by an arrow in $G$’ condition. This requires that every other point in $||[*/1]||$ must also map to the origin under $f_0$. The map on objects is

$$f_0 : ||[*/1]|| \to D^2;$$
$$x \mapsto 0.$$  

When it comes to what the functor does to the arrows, note that the isotropy group at 0 in $[D^2/\frac{Z}{3}]$ is $\frac{Z}{3}$. Therefore, any arrow $x \leftarrow x'$ on the left-hand side must map to a $0 \leq 0$ on the right-hand side, where $g \in \{id, a, a^2\} = \frac{Z}{3}$. However, it was said that $||[*/1]||$ is an infinite dimensional contractible space, and that there is a unique arrow between any two objects. By contractibility, $x'$ can be moved continuously to $x$, which in turn induces a continuous map from the arrow $x \leftarrow x'$ to the arrow $x \leftarrow x$ (using the uniqueness of the arrow from $x$ to $x$). Since the image of the identity arrow must be the identity arrow, this continuous map on arrows must be reflected by a continuous map of arrows from $0 \leq 0$ to $0 \leq 0$ on the right-hand-side. Using the fact that $\frac{Z}{3}$ is a discrete group, this is only possible in the case that $g = id$. This process can be repeated at every $x \in ||[*/1]||$, which shows that the map on arrows is

$$f_1 : \{x \leftarrow x' | x, x' \in ||[*/1]||\} \to \{0 \leq 0 | g \in \frac{Z}{3}\},$$
$$(x \leftarrow x') \mapsto (0 \leftarrow 0).$$

The conclusion is that if there is any $x \in ||[*/1]||$ such that $f_0(x) = 0 \in D^2$ then the only functor $fib([*/1]) \to [D^2/\frac{Z}{3}]$ is

$$f : fib([*/1]) \to [D^2/\frac{Z}{3}],$$
$$x \mapsto 0,$$
$$(x \leftarrow x') \mapsto (0 \leftarrow 0).$$

In the second case, the assumption is that there is an $x \in ||[*/1]||$ such that $f_0(x) \neq 0$. As before, invoke the ‘every two points in the image must be connected by an arrow in $G$’ condition. In this case, the requirement becomes that if $x' \in ||[*/1]||$ then,

$$f_0(x') \in \{f_0(x), e^{2\pi i/3}f_0(x), e^{4\pi i/3}f_0(x)\}.$$

Using the condition that the map $f_0$ is continuous, and noting that the set of allowed values for $f_0(x')$ is discrete, further restricts the value of $f_0(x')$ so that it must in fact be equal to $f_0(x)$. Since $f_0(x) \neq 0$, the only arrow $f_0(x) \leftarrow f_0(x)$
is the identity arrow.

This leads to the result that if there exists $x \in |[*/1]|$ such that $f_0(x) \neq 0$, then the only functor $\text{fib}([*/1]) \to [D^2/\mathbb{Z}_3]$ is

$$f : \text{fib}([*/1]) \to [D^2/\mathbb{Z}_3],$$

$$y \mapsto f_0(x),$$

$$(y \leftarrow y') \mapsto (f_0(x) \leftarrow f_0(x)).$$

In both cases, a functor is uniquely defined by picking its value at a single point. Putting both together shows that the space of functors is

$$\text{map}(\text{fib}([*/1]), [D^2/\mathbb{Z}_3]) \cong D^2.$$

After working through the above example, it should be clear that this argument will work when $[D^2/\mathbb{Z}_3]$ is replaced by any topological groupoid $[M/G]$, where $M$ is a topological space and $G$ is a discrete group which acts on $M$.

**Theorem 2.4.1.** Let $M$ be a space and let $G$ be a discrete group which acts on $M$. Then

$$\text{map}([*/1], [M/G]) \cong M.$$

To emphasise the point, the important assumption here is that $G$ is discrete; the above argument will not work for more general groups. Specifically, this assumption does not hold for the fibrant replacement of a topological groupoid, greatly restricting the usefulness of this example. However, it does give a feel for these type of objects.

**Example 2.4.2.** Consider another example of the mapping space:

$$\text{map}(\text{fib}([*/1]), \text{fib}(\mathcal{G})).$$

First, invoke a proposition of Gepner-Henriques [GH17, Proposition 2.52] which proves that

$$\text{map}(\text{fib}([*/1]), \text{fib}(\mathcal{G})) \simeq \text{map}([*/1], \text{fib}(\mathcal{G})).$$

If one is only interested in the homotopy type of the mapping space, then the fibrant replacement in the first factor can be ignored.

The mapping space $\text{map}([*/1], \text{fib}(\mathcal{G}))$ is quite easy to describe. Each element of the mapping space consists of

1. a map on objects $* \to |\mathcal{G}|$; and
2. a map on arrows.

As to not get bogged down in the arrows map, see that since the only arrow on the left-hand side is the identity arrow, the arrow must map to the identity arrow at the image of \(*\) on the right-hand side. An element of the mapping space is therefore uniquely determined by the choice for the image of \(*\). Furthermore, every point in \(\|\mathcal{G}\|\) is a valid choice.

The result is that this mapping space has an easy to understand homotopy type:

\[ \text{map}(\text{fib}(\{\ast\}/1), \text{fib}(\mathcal{G})) \simeq \|\mathcal{G}\|, \]

the fat geometric realisation of \(\mathcal{G}\).

Consider now the space of natural transformations,

\[ \text{Nat}(R(\mathcal{G}), R(\mathcal{H})). \]

To see how this relates to easier to understand objects, observe the map from the space of continuous functors,

\[ \text{map}(\text{fib}(\mathcal{G}), \text{fib}(\mathcal{H})) \to \text{Nat}(R(\mathcal{G}), R(\mathcal{H})), \]

\[ g \mapsto ((f : \text{fib}(\{\ast\}/K)) \to \text{fib}(\mathcal{G})) \mapsto (g \circ f : \text{fib}(\{\ast\}/K) \to \text{fib}(\mathcal{H}))). \]

There is also a map

\[ \text{Nat}(R(\mathcal{G}), R(\mathcal{H})) \to \text{map}(\|\mathcal{G}\|, \|\mathcal{H}\|), \]

given by evaluating the natural transformations at the trivial group, \(1\) (Recalling that \(R(\mathcal{G})(1) = \text{map}(\text{fib}(\{\ast\}/1), \text{fib}(\mathcal{G})) \simeq \|\mathcal{G}\|)).

The composition of these maps,

\[ \text{map}(\text{fib}(\mathcal{G}), \text{fib}(\mathcal{H})) \to \text{map}(\|\mathcal{G}\|, \|\mathcal{H}\|) \]

is a map that in a perfect world, would be a homotopy equivalence. This section concludes with some evidence as to why the author believes this could be the case.

Assuming that \(\text{map}(\|\mathcal{G}\|, \|\mathcal{H}\|)\) and \(\|\text{Map}(\mathcal{G}, \mathcal{H})\|\) are equivalent, one can attempt to build a homotopy inverse using a map

\[ \|\text{Map}(\mathcal{G}, \mathcal{H})\| \to \text{map}(\mathcal{G}, \text{fib}(\mathcal{H})), \]

which is defined as follows. An element of \(\|\text{Map}(\mathcal{G}, \mathcal{H})\|\) is of the form \((r_0 \cdot \varphi_0 \cdot g_1)\).
2.4. EXAMPLES RELATING TO THE FUNCTOR \( R(\mathcal{G}) \)

\[ \ldots \xleftarrow{g_n} r_n \cdot \varphi_n \] where \((r_0, \ldots, r_n) \in \Delta^n\), \(\varphi_i\) is a functor \(\mathcal{G} \to \mathcal{H}\) and \(g_i\) is a natural transformation \(\varphi_i \Rightarrow \varphi_{i-1}\). The image of \((r_0 \cdot \varphi_0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot \varphi_n)\) is the continuous functor,

\[ \mathcal{G} \to fib(\mathcal{H}), \]

\[ x \mapsto (r_0 \cdot \varphi_0(x) \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot \varphi_n(x)) \text{ on objects, and} \]

\[ (x \xleftarrow{h} x') \mapsto (r_0 \cdot \varphi_0(x) \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot \varphi_n(x) \xleftarrow{\varphi_0(h)} r_0 \cdot \varphi_0(x') \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot \varphi_n(x')) \text{ on arrows.} \]

It is shown in [GH07] that this functor is an isomorphism in the case that \(\mathcal{G} = [*/\mathcal{G}]\) and \(\mathcal{H} = [*/\mathcal{H}]\).

Having the isomorphism case for the \([*/\mathcal{G}]\) cases, now consider a slightly more complicated case where the object space of the orbispaces are no-longer just a single point.

**Example 2.4.3.** This example shows that \(|\text{Map}(\mathcal{G}, \mathcal{H})|\) and \(\text{map}(\text{fib}(\mathcal{G}), \text{fib}(\mathcal{H}))\) are homotopy equivalent when \(\mathcal{G} = [*/\frac{\pi}{2}]\) and \(\mathcal{H} = [D^2/\frac{\pi}{2}]\), where \(\frac{\pi}{2}\) acts on \(D^2\) by rotation.

Begin by looking at the space \(|\text{Map}([*/\frac{\pi}{2}], [D^2/\frac{\pi}{2}])|\). Recall that the object space of \(\text{Map}([*/\frac{\pi}{2}], [D^2/\frac{\pi}{2}])\) is the space of continuous functors, written \(\text{map}([*/\frac{\pi}{2}], [D^2/\frac{\pi}{2}])\). Such a functor, \(F\), is a continuous map on both objects and morphims. On objects, the functor simply assigns a point \(F(*) \in D^2\). As for what \(F\) does to the arrow space, \(\frac{\pi}{2}\), there are two possibilities.

1. If \(F(*) \neq 0 \in D^2\), then there is only one choice for the image of \(\frac{\pi}{2}\) in \([D^2/\frac{\pi}{2}]\). Note that there are only two arrows on the right-hand side with source \(F(*)\), namely the identity arrow, and the rotation by \(\pi\) arrow. As always, the identity arrow on the left must map to the identity arrow on the right. Thinking of the group as \(\frac{\pi}{2} = \{id, \alpha\}\) with \(\alpha^2 = id\) one can see that \(F(\alpha) = id\), the identity arrow at \(F(*)\). Indeed, it is required that \(s(F(\alpha)) = t(F(\alpha)) = F(*)\), and the only such map when \(F(*) \neq 0\) is the identity arrow at \(F(*)\). In this case the functor is

\[ F : [*/\frac{\pi}{2}] \to [D^2/\frac{\pi}{2}], \]

\[ * \mapsto F(*) \neq 0, \]

\[ id \mapsto id, \]

\[ \alpha \mapsto id. \]

2. If \(F(*) = 0 \in D^2\), then there are two possibilities for the image of \(\frac{\pi}{2}\) in
In this case both the arrows on the right-hand side have source and target equal to 0. As with the previous case, the identity arrow on the left must map to the identity on the right. However, in contrast with the first case, it is possible to map the non-trivial element \( \alpha \) on the left to either the identity arrow at 0 or the arrow \( \alpha \) at 0 on the right. There are therefore two possible functors with \( F(*) = 0 \):

\[
F_1 : \left[ \ast / \frac{\pi}{2} \right] \to \left[ D^2 / \frac{\pi}{2} \right], \\
F_2 : \left[ \ast / \frac{\pi}{2} \right] \to \left[ D^2 / \frac{\pi}{2} \right],
\]

\[
\begin{align*}
* & \mapsto 0, & * & \mapsto 0, \\
\text{id} & \mapsto \text{id}, & \text{id} & \mapsto \text{id}, \\
\alpha & \mapsto \text{id}, & \alpha & \mapsto \alpha.
\end{align*}
\]

One should also note that the \( F_1 \) functor above fits into the functors of case 1, filling in the ‘hole’ at \( F(*) = 0 \).

The next step is to describe the arrow space of \( \text{Map} \left( \left[ \ast / \frac{\pi}{2} \right], \left[ D^2 / \frac{\pi}{2} \right] \right) \) which is natural transformations between these functors. In this case, the picture of a natural transformation \( \eta : F \Rightarrow G \) is two commutative diagrams:

\[
\begin{array}{ccc}
F(*) & \xrightarrow{F(\text{id})} & F(*) \\
\downarrow \eta_* & & \downarrow \eta_* \\
G(*) & \xrightarrow{G(\text{id})} & G(*)
\end{array}
\quad
\begin{array}{ccc}
F(*) & \xrightarrow{F(\alpha)} & F(*) \\
\downarrow \eta_* & & \downarrow \eta_* \\
G(*) & \xrightarrow{G(\alpha)} & G(*)
\end{array}
\]

Recall from above that there are only two arrows beginning at \( F(*) \), namely the identity arrow, \( \text{id} \), and the rotation by \( \pi \) arrow, \( \alpha \), which are the allowed values of \( \eta_* \). As before, there are two cases to consider, the \( F(*) \neq 0 \in D^2 \) case and the \( F(*) = 0 \in D^2 \) case.

1. If \( F(*) \neq 0 \), then it has been shown above that \( F(\text{id}) = \text{id} \) and \( F(\alpha) = \text{id} \).

The vertical maps can either be \( \eta_* = \text{id} \) or \( \eta_* = \alpha \). The \( \eta_* = \text{id} \) case gives \( G(*) = F(*) \) with \( G(\text{id}) = \text{id} \) and \( G(\alpha) = \text{id} \). The \( \eta_* = \alpha \) case gives \( G(*) = -F(*) \) (where \( -F(*) \) is used to denote \( F(*) \) rotated by \( \pi \)) with \( G(\text{id}) = \text{id} \) and \( G(\alpha) = -\text{id} \). Note that it is not possible for \( G(\alpha) = \alpha \) here, as \( G(*) \neq 0 \) (since \( F(*) \neq 0 \)). The two commutative diagrams both become the same and one obtains two natural transformations with source \( F \):

\[
\begin{array}{ccc}
F(*) & \xrightarrow{\text{id}} & F(*) \\
\downarrow \text{id} & & \downarrow \text{id} \\
F(*) & \xrightarrow{\text{id}} & F(*),
\end{array}
\quad
\begin{array}{ccc}
F(*) & \xrightarrow{\text{id}} & F(*) \\
\downarrow \alpha & & \downarrow \alpha \\
-F(*) & \xrightarrow{\text{id}} & -F(*).
\end{array}
\]
2.4. EXAMPLES RELATING TO THE FUNCTOR \( R(\mathcal{G}) \)

2. If \( F(*) = 0 \), then \( F(id) = id \) and \( F(\alpha) \in \{id, \alpha\} \). The vertical maps can either by \( \eta_* = id \) or \( \eta_* = \alpha \). Both of these cases give \( G(*) = F(*) = 0 \). The two commutative diagrams give two natural transformations in the case that \( F(\alpha) = id \) and two natural transformations in the case that \( F(\alpha) = \alpha \).

At this point one can see that there is a single functor when \( F(*) \neq 0 \) and there is a natural transformation to the functor at \( -F(*) \). There are two functors at \( F(*) = 0 \), though they are not connected by a natural transformation. However, each of these two functors is connected to itself by two natural transformations (corresponding to \( id \) and \( \alpha \)). Again note that the \( F(*) = 0, F(\alpha) = id \) case corresponds to the limit of the \( F(*) \neq 0 \) case as \( F(*) \to 0 \).

Having described both the objects and morphisms it is now possible to build the classifying space \( ||\text{Map}(\star/\Delta^2, [D^2/\Delta^2])|| \). Recall that a point in \( ||\text{Map}(\star/\Delta^2, [D^2/\Delta^2])|| \) is made up of a point in \( \Delta^n \) and a chain of \( n \) natural transformations, along with the standard identifications. In this case, a point is a choice of point in \( \Delta^n \), a functor in \( \text{map}(\star/\Delta^2, [D^2/\Delta^2]) \), and an element of the set \( \{id, \alpha\}^\times n \). This space is made up of two disjoint parts:

1. The first part is made up of the \( F(\alpha) = id \) functors. As described above, such a functor is uniquely defined by a choice of \( F(*) \in D^2 \). The \( F(\alpha) = id \) part of the classifying space is therefore

\[
\{id, \alpha\}^\times n \times D^2 \times \Delta^n,
\]
with the classifying space identifications.

2. The remaining part is made up of the $F(\alpha) = \alpha$ functor. As described above, this functor only exists when $F(\ast) = 0 \in D^2$. The $F(\alpha) = \alpha$ part of the classifying space is

$$\{id, \alpha\}^n \times \{0\} \times \Delta^n,$$

with the classifying space identifications.

Putting these cases together gives the classifying space,

$$\|Map([\ast/\frac{\pi}{2}], [D^2/\frac{\pi}{2}])\| = \bigsqcup_n \{id, \alpha\}^n \times (D^2 \sqcup \{0\}) \times \Delta^n/\sim.$$

The other half of the problem is to investigate the mapping space of functors

$$\text{map}(\text{fib}([\ast/\frac{\pi}{2}]), \text{fib}([D^2/\frac{\pi}{2}])).$$

This space can be simplified, invoking [GHU], Proposition 2.52] gives a homotopy equivalence

$$\text{map}(\text{fib}([\ast/\frac{\pi}{2}]), \text{fib}([D^2/\frac{\pi}{2}])) \simeq \text{map}([\ast/\frac{\pi}{2}], \text{fib}([D^2/\frac{\pi}{2}])).$$

Suppose $F$ is a functor from $[\ast/\frac{\pi}{2}]$ to $\text{fib}([D^2/\frac{\pi}{2}])$. On objects, $F$ is a choice of point $F(\ast) \in \| [D^2/\frac{\pi}{2}] \|$. That is,

$$F(\ast) = r_0 \cdot x_0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot x_n,$$

where $g_i \in \{id, \alpha\} = \frac{\pi}{2}$, $x_i \in D^2$ and $r \in \Delta^n$. Note that since the only arrows are the identity and the rotate by $\pi$ arrow, $x_1 \in \{x_0, -x_0\}$, where $-x_0$ is $x_0$ rotated by $\pi$. Furthermore, an arrow in $\text{fib}([D^2/\frac{\pi}{2}])$ is of the form

$$r_0 \cdot x_0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot x_n \xleftarrow{\gamma} s_0 \cdot y_0 \xleftarrow{h_1} \cdots \xleftarrow{h_m} s_m \cdot y_m,$$

where $\gamma \in \{id, \alpha\}$, which implies that $y_0 \in \{x_n, -x_n\}$.

On the left-hand side, $s(id) = t(id) = s(\alpha) = t(\alpha) = \ast$. Therefore it is the case on the right-hand side that $s(F(\alpha)) = t(F(\alpha)) = F(\ast)$. As with the classifying space before, there are two cases

1. If $F(\ast) = r_0 \cdot x_0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot x_n$ with every $x_i \neq 0 \in D^2$, then the only arrow with both source and target at $F(\ast)$ is the identity arrow at $F(\ast)$. Therefore, such a functor is uniquely defined by the choice of
2.4. EXAMPLES RELATING TO THE FUNCTOR \( R(\mathcal{G}) \)

\[ F(*) \in \| [D^2/\Delta^2] \|, \]

\[ F : [\ast/\Delta^2] \to [D^2/\Delta^2], \]

\[ * \mapsto F(*), \]

\[ id \mapsto id_{F(*)}, \]

\[ \alpha \mapsto id_{F(*)}. \]

2. If \( F(*) = r_0 \cdot 0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot 0 \) there are two arrows which both have source and target \( F(*) \). These two arrows are the identity arrow at \( F(*) \),

\[ r_0 \cdot 0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot 0 \xleftarrow{(g_1 \cdots g_n)^{-1} \circ \text{id}} r_0 \cdot 0 \xleftarrow{h_1} \cdots \xleftarrow{h_n} r_n \cdot 0, \]

and the arrow coming from the arrow \( \alpha \) in \([D^2/\Delta^2]\),

\[ r_0 \cdot 0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot 0 \xleftarrow{(g_1 \cdots g_n)^{-1} \circ \alpha} r_0 \cdot 0 \xleftarrow{h_1} \cdots \xleftarrow{h_n} r_n \cdot 0. \]

Call these two arrows \( id \) and \( \pi \) respectively. As observed in case 1, taking \( F(*) = id \) gives a functor

\[ F_1 : [\ast/\Delta^2] \to [D^2/\Delta^2], \]

\[ * \mapsto r_0 \cdot 0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot 0, \]

\[ id \mapsto id, \]

\[ \alpha \mapsto id. \]

Another functor exists only at \( F(*) = 0 \) and is

\[ F_2 : [\ast/\Delta^2] \to [D^2/\Delta^2], \]

\[ * \mapsto r_0 \cdot 0 \xleftarrow{g_1} \cdots \xleftarrow{g_n} r_n \cdot 0, \]

\[ id \mapsto id, \]

\[ \alpha \mapsto \pi. \]

Mirroring the process taken with the classifying space, earlier, the functors are in two disjoint components.

1. The \( F(*) = \text{id} \) functors are uniquely determined by the choice of point in \( \| [D^2/\Delta^2] \| \). Such a point is made up of a point \( r \in \Delta^n \), a point \( x_0 \in D^2 \), and an element of the set \( \{id, \alpha\}^\times n \). This space of functors is

\[ \{id, \alpha\}^\times n \times D^2 \times \Delta^n, \]
with the classifying space identifications.

2. The $F(\alpha) = \pi$ case only exists when $x_i = 0 \in D^2$. The point $F(*)$ is made up of a point $r \in \Delta^n$, and an element of the set $\{id, \alpha\}^{\times n}$, with $x_0 = 0$. This space of functors is

$$\{id, \alpha\}^{\times n} \times \{0\} \times \Delta^n,$$

with the classifying space identifications.

Putting both of these cases together gives the mapping space,

$$\text{map}(\ast / \frac{\mathbb{Z}}{2} \sqcup \text{fib}(\ast / \frac{\mathbb{Z}}{2}^2)) = \bigcup_n \{id, \alpha\}^{\times n} \times (D^2 \sqcup \{0\}) \times \Delta^n / \sim,$$

the same description as for $\|\text{Map}(\ast / \frac{\mathbb{Z}}{2} \sqcup D^2 / \frac{\mathbb{Z}}{2})\|/\sim$.

Therefore,

$$\text{map}(\ast / \frac{\mathbb{Z}}{2} \sqcup \text{fib}(D^2 / \frac{\mathbb{Z}}{2})) \simeq \|\text{Map}(\ast / \frac{\mathbb{Z}}{2} \sqcup D^2 / \frac{\mathbb{Z}}{2})\|/\sim.$$

As a final note with this example, it is possible to slightly simplify the description of these spaces. Noting that $\sqcup_n \{id, \alpha\}^{\times n} \times \Delta^n / \sim$ is precisely the fat geometric realisation of $\frac{\mathbb{Z}}{2}$, the spaces above are homotopy equivalent to:

$$\|\frac{\mathbb{Z}}{2}\| \times (D^2 \sqcup \{0\}).$$
Chapter 3

Homological Injectivity for Configurations on Orbifolds

Presented in this chapter is a proof of homological injectivity for configurations on orbifolds. An important assumption for this chapter is that the orbifold in question, $\mathcal{X}$, must be the interior of an orbifold with boundary, and it should look like $\partial \mathcal{X} \times I$ near the boundary. This assumption can be weakened to only requiring the existence of a full, closed sub-orbifold $\mathcal{U} \subseteq \partial \mathcal{X}$ so that $\mathcal{X}$ looks like $\mathcal{U} \times I$ near $\mathcal{U}$.

The form of this proof is analogous to that for manifolds which can be found in $\cite{McD75, RW11}$. The first step is to define a stabilisation map, which adds a point to a configuration on the orbifold. This induces a map on the homology of the configuration spaces, which is then shown to be injective.

3.1 The Stabilisation Map

Let $\mathcal{X}$ be the interior of an orbifold with boundary (Definition $\cite{McD75, RW11}$) such that there is a neighbourhood of $\partial \mathcal{X}$ which is homeomorphic to $\partial \mathcal{X} \times I$.

In order to add a new point to a configuration on $\mathcal{X}$, one needs to have a self-map $\mathcal{X} \to \mathcal{X}$ which ‘pushes away’ configuration points from an area, leaving a place to add a new point. This process is done near the boundary of $\mathcal{X}$ and is often called ‘adding a point at infinity’ to the configuration. Since the orbifold is homeomorphic to $\partial \mathcal{X} \times I$ near the boundary, construct a collar of $\mathcal{X}$. Such a collar is an embedding,

$$j : \partial \mathcal{X} \times I \to \mathcal{X},$$

with the requirement that $j$ restricted to the boundary, $j|_{\partial \mathcal{X} \times \{0\}} : \partial \mathcal{X} \hookrightarrow \mathcal{X},$
is the standard inclusion. This embedding is used to construct a map which pushes points away from the boundary and gives a place to add a point to a configuration on $\mathcal{X}$. Such a map which pushes points away is,

$$k : \overline{\mathcal{X}} \to \overline{\mathcal{X}},$$

$$x \mapsto \begin{cases} x, & \text{if } x \in \overline{\mathcal{X}} \setminus \text{im}(j), \\ j(\hat{k}(j^{-1}(x))), & \text{if } x \in \text{im}(j), \end{cases}$$

where,

$$\hat{k} : \partial \overline{\mathcal{X}} \times I \to \overline{\mathcal{X}},$$

$$(d,t) \mapsto (d, \frac{1}{2} + \frac{t}{2}).$$

This map pushes points into the bottom ‘half’ of the collar around the boundary, and the image $k(\overline{\mathcal{X}})$ of any point will end up in the interior of $\overline{\mathcal{X}}$. To add a point to a configuration on $\mathcal{X}$, one does so by adding a point on the boundary of $\overline{\mathcal{X}}$ followed by using the map $k$ to push it into $\mathcal{X}$ proper. The result is a configuration on $\mathcal{X}$ with one more point than before. The assumption here that there is a collar neighbourhood around the entire boundary of $\overline{\mathcal{X}}$ is quite restrictive. It can be weakened to only assuming that there is a collar around a full sub-orbifold of $\partial \overline{\mathcal{X}}$.

**Definition 3.1.1.** Let $\mathcal{X}$ be an orbifold. An orbifold $\mathcal{Y}$ is called a full sub-orbifold of $\mathcal{X}$ if

1. the spaces $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are sub-manifolds of the spaces $\mathcal{X}_0$ and $\mathcal{X}_1$; and

2. if $y \in \mathcal{Y}_0$ and $g \in \mathcal{X}_1$ such that $s(g) = y$ then $g \in \mathcal{Y}_1$ and $t(g) \in \mathcal{Y}_0$.

Let $\mathcal{U}$ be the interior of a closed full sub-orbifold of $\partial \overline{\mathcal{X}}$ and define $\mathcal{V}$ to be

$$\mathcal{V} := \frac{\mathcal{U} \times I}{\text{if } u \in \text{obj}(\partial \mathcal{U}) \text{ and } t,t' \in I \text{ then } (u,t) \sim (u,t')}.$$ 

Assume that there is an orbifold embedding

$$j : \mathcal{V} \to \overline{\mathcal{X}},$$

such that $j|_{\mathcal{U} \times \{0\}} : \mathcal{U} \hookrightarrow \overline{\mathcal{X}}$ is the standard inclusion. Note that $j$ being an embedding says that $j|_{\mathcal{U} \times I}$ is injective. See Figure 3.1 for a picture of the collar of a sub-orbifold of $\partial \overline{\mathcal{X}}$.

Using this new $j$, one constructs a comparable map to the $k$ above (which
3.1. THE STABILISATION MAP

Figure 3.1: \( V \) forms a collar of a full sub-orbifold of \( \partial \mathcal{X} \).

Figure 3.2: The map \( k \) pushes the points in the image of \( j \) away from the boundary, giving a place to add a point to a configuration on \( \mathcal{X} \).

was for a collar around the entire boundary \( \partial \mathcal{X} \). Define the map on objects,

\[
k : \mathcal{X} \to \mathcal{X},
\]

\[
x \mapsto \begin{cases} x, & \text{if } x \in \mathcal{X} \setminus j(\mathcal{U} \times I), \\ j(\hat{k}(j^{-1}(x))), & \text{if } x \in j(\mathcal{U} \times I), \end{cases}
\]

where,

\[
\hat{k} : \mathcal{V} = (\mathcal{U} \times I)/ \sim \to \mathcal{V},
\]

\[
(d, t) \mapsto (d, \frac{1}{2} + \frac{t}{2}).
\]

Note that since \( j|_{\mathcal{U} \times I} \) is injective, a unique inverse point, \( j^{-1}(x) \), exists for every \( x \in j(\mathcal{U} \times I) \), which means that \( j(\hat{k}(j^{-1}(x))) \) consists of a single point (this is not true for \( x \in j(\partial \mathcal{U} \times I) \)). The map \( k \) is isotopic to the identity. For a picture of what \( k \) does to the image of \( j \), see Figure 3.2. Importantly, one can see that if \( x \in \mathcal{U} \), then \( x \notin k(\mathcal{X}) \).
CHAPTER 3. HOMOLOGICAL INJECTIVITY

Figure 3.3: The stabilisation map, $s_\varepsilon$, on a configuration $\vec{x} \in \text{Conf}_3(\mathcal{X})$ is made up of adding a point $\varepsilon \in \mathcal{U}$, followed by pushing everything in $\mathcal{V}$ away from the boundary. The result is an element in $\text{Conf}_4(\mathcal{X})$.

Define the stabilisation map at a point $\varepsilon \in \mathcal{U}$ to be,

$$s_\varepsilon : \text{Conf}_n(\mathcal{X}) \rightarrow \text{Conf}_{n+1}(\mathcal{X}),$$

$$\vec{x} \mapsto k(\vec{x} \cup \{\varepsilon\}).$$

See Figure 3.3 for a picture of how this works. As $\varepsilon$ is not a point in the configuration $s_\varepsilon(\vec{x})$ for any $\vec{x}$, this process can be iterated as many times as needed.

3.2 The Transfer Map

Recall the definition of a covering space.

**Definition 3.2.1 (Covering Space).** Let $Y$ be a topological space. A covering space of $Y$ is a topological space $X$ with a surjective map $p : X \rightarrow Y$ such that if $y \in Y$ then there exists an open neighbourhood $U_y \subseteq Y$ satisfying

1. $y \in U_y$,
2. $p^{-1}(U_y)$ is the disjoint union of open sets in $X$, and
3. if $V$ is a connected component of $p^{-1}(U_y)$ then $p$ maps $V$ homeomorphically onto $U_y$.

A covering space is said to be of degree $d$ if the second condition becomes $p^{-1}(U_y)$ is a disjoint union of $d$ open sets in $X$.

Let $p : X \rightarrow Y$ be a finite sheeted cover of degree $d$. It is common to look at the map induced on homology by $p$,

$$p_* : H_*X \rightarrow H_*Y.$$

There is also an induced map in the opposite direction, the transfer map,

$$p^! : H_*Y \rightarrow H_*X,$$
3.2. THE TRANSFER MAP

which is an important map used in many homological stability proofs.

Let \( U = \{ U_\alpha \subseteq Y \mid \alpha \in \mathcal{A} \} \) be a cover of \( Y \) be such that \( p^{-1}U_\alpha \cong U_\alpha \uplus^d \), with \( \mathcal{A} \) an indexing set. The transfer map is defined via the chain complex of \( Y \). If \( C_*Y \) is the chain complex of \( Y \), define

\[
C^d_*Y := \{ s \in C_*Y \mid \text{there exists } \alpha \in \mathcal{A} \text{ such that } s \subseteq U_\alpha \},
\]

that is, all the simplices in \( C_*Y \) which are contained entirely within a \( U_\alpha \). The inclusion map \( C^d_*Y \hookrightarrow C_*Y \) induces an isomorphism on homology (see [Hat02, Proposition 2.21]). For \( n \in \mathbb{Z}_{\geq 0} \), there is a diagram,

\[
p^{-1}U_\alpha \xrightarrow{=} U_\alpha \uplus^d \xrightarrow{f_1, \ldots, f_d} \Delta^n
\]

where \( f_1, \ldots, f_d \) are the \( d \) possible liftings of \( f(\Delta^n) \) into \( p^{-1}U_\alpha \).

The map on simplices is

\[
C^d_*Y \rightarrow C_*X,
\]

\[
f \mapsto \sum_{i=1}^d f_i.
\]

The map induced on homology by this map on chain complexes is the transfer map of \( p \),

\[
p^! : H_*Y \rightarrow H_*X.
\]

The transfer map, \( p^! \), has a couple of interesting properties. Firstly, if \( \alpha \in H_*Y \), then applying the transfer map, followed by the covering map multiplies by the degree, \( d \),

\[
p_*p^! \alpha = d\alpha.
\]

Composing the induced maps in the opposite direction gives something less concrete. If \( \beta \in H_*X \), then it is in general difficult to describe \( p^!p_*\beta \). Intuitively this object should be all the parts of \( H_*X \) which lie in the same fibre as \( \beta \). This can be made more precise in the case that \( G \) is a discrete group acting on
covering maps. If $p$ is a normal covering space with Galois group $G$, then

$$p^* p_* \beta = \sum_{g \in G} g_* \beta.$$ 

The following proposition is considered by some to be their “favourite trick ever” [Wes12].

**Proposition 3.2.2.** Let $p: X \to Y$ be a covering of degree $d$. If $H_n(X, \mathbb{Q}) = 0$, then $H_n(Y, \mathbb{Q}) = 0$.

**Proof.** To see why this is the case, look at the composition,

$$H_n(Y, \mathbb{Q}) \xrightarrow{p^*} H_n(X, \mathbb{Q}) \xrightarrow{p_*} H_n(Y, \mathbb{Q}).$$

The assumption is that $H_n(X, \mathbb{Q}) = 0$. From above, this composition of maps is multiplication by $d$,

$$p_* p^* \alpha = d \alpha.$$ 

Multiplication by $d$ is injective and, since $d$ is invertible in $\mathbb{Q}$, has inverse map $\frac{1}{d}$. The only way this is possible is when there is only one element of $H_n(Y, \mathbb{Q})$. Thus $H_n(Y, \mathbb{Q}) = 0$. \qed

As mentioned above, these types of constructions, using transfer-like maps, appear in a number of areas. A transfer product in cohomology of symmetric groups is defined in [ST97], which is then used to calculate the mod-2 cohomology of all symmetric groups in [GSS12]. Another place where similar constructions are used is in [Knu15] to calculate the Betti numbers for configuration spaces via factorisation homology.

### 3.3 The Map $Bp : BConf_{n,m}\mathcal{X} \to BConf_n\mathcal{X}$ is a Cover Up To Homotopy

In addition to the unordered configuration space $Conf_n\mathcal{X}$, define the unordered configuration space on $n$ points with grouped points. The space $Conf_{n,m}\mathcal{X}$ is used to denote configurations of $n$ points on $\mathcal{X}$ with each of the $n$ points grouped into one of two sets, one of size $m$ and one of size $n - m$. There is an obvious map

$$p : Conf_{n,m}\mathcal{X} \to Conf_n\mathcal{X},$$

which simply forgets the groupings of the points in a configuration.
Recall that the orbifold $Conf_n(X)$ can be thought of as having

1. object space

\[ \{(x_1, \ldots, x_n) \in X^n_0 \mid \text{if } i \neq j \text{ then there is no arrow } x_i \to x_j \}, \]

which are ordered tuples of points which do not hit any other point or any of their ghost points; and

2. arrow space made from a reordering of points (an element of $S_n$), and an arrow from $X_1$ for each point in the configuration. Two elements of the configuration space are connected by an arrow if they differ by a reordering of the points or if a point moves to one of its ghost points. More precisely, there is an arrow $\bar{x}^1 \to \bar{x}^2$ if there exists $\sigma \in S_n$ and $\gamma_1, \ldots, \gamma_n \in X_1$ such that

\[ \gamma_i : x^1_i \to x^2_{\sigma(i)}, \quad i \in \{1, \ldots, n\}. \]

There is a similar description of the orbifold $Conf_{n,m}(X)$. One can think of $Conf_{n,m}(X)$ as the orbifold with,

1. object space as above for $Conf_n(X)$, ordered $n$-tuples of points in $X_0$ in which no point is equal to another, or any of their ghost points; and

2. arrow space similar to above but with $\sigma \in S_m \times S_{n-m} \subseteq S_n$. One can see that restricting $\sigma$ to being in $S_m \times S_{n-m}$ only allows re-ordering within the first $m$ co-ordinates and within the last $n-m$ co-ordinates, effectively grouping the points into a set of size $m$ and a set of size $n-m$.

As stated above, the map $p : Conf_{n,m}X \to Conf_nX$ is intuitively forgetting the $m$ and $n-m$ groupings. This corresponds to the orbifold map with,

1. map on the objects the identity,

\[ p_0 : \text{obj}(Conf_{n,m}X) \to \text{obj}(Conf_nX), \]

\[ \bar{x} \mapsto \bar{x}, \]

and

2. map on arrows made up of the identity on the $X_1$ part and the inclusion
of $S_m \times S_{n-m}$ into $S_n$ on the re-ordering part,

$$p_1 : \text{mor}(\text{Conf}_{n,m}X) \to \text{mor}(\text{Conf}_nX),$$

$$p_1 : \text{mor}(P\text{Conf}_nX) \times (S_{n-m} \times S_m) \leftrightarrow \text{mor}(P\text{Conf}_nX) \times S_n,$$

$$(f, \sigma) \mapsto (f, \sigma).$$

This map $p : \text{Conf}_{n,m}X \to \text{Conf}_nX$ induces a map on the classifying spaces $Bp : B\text{Conf}_{n,m}X \to B\text{Conf}_nX$. The aim of this section is to show that this map, $Bp$, is a cover map up to homotopy. To do this, one needs to check that the fibre of $Bp$ is homotopically discrete, that is, the fibre is a union of components, each of which is contractible. The argument below uses a version of Quillen’s Theorem B, modified for use with topological categories. Quillen’s Theorem B uses comma categories, the definition of which will be as presented in [Wei13].

**Definition 3.3.1.** Let $C$ and $D$ be categories and let $F : C \to D$ be a functor. Let $d \in D$. The **comma category** $d \downarrow F$ is the category with,

- objects,

$$\{(c, f) \in \text{obj}(C) \times \text{mor}(D) \mid f : d \to F(c)\},$$

and,

- a morphism from $(c_1, f_1)$ to $(c_2, f_2)$ is a morphism $h \in \text{Hom}_C(c_1, c_2)$ such that,

$$\begin{array}{c}
\downarrow d \\
F(c_1) \quad F(h) \quad F(c_2) \\
\downarrow f_1 \quad \downarrow f_2
\end{array}$$

commutes in $D$.

The topologically enriched version of Quillen’s Theorem B, below, is a consequence of the main theorem from [Mey84].

**Theorem 3.3.2.** Assume that if $b : B' \to B$ is an arrow in $\text{Conf}_nX$, then $b$ induces a homotopy equivalence

$$B(B \setminus p) \xrightarrow{\simeq} B(B' \setminus p).$$

Then $Bp$ has homotopy fibre $B(B \setminus p)$.

---

1 The author realises there are a lot of occurrences of the letter $B$ in this theorem. For the confused reader, from left-to-right in $B(B \setminus p) \simeq B(B' \setminus p)$, the $B$s in order are: Classifying space, target of $b$, homotopy equivalence induced by $b$ (hidden), classifying space, source of $b$. 
The reader who is examining Meyer’s original reference should observe that the spaces $\text{Conf}_nX$ and $\text{Conf}_{n,m}X$ are indeed ‘good’. This is due to the fact that the identity elements in the arrow space are non-degenerate base-points.

If the assumption is satisfied, this theorem says that

$$\text{hofib}(B\text{Conf}_{n,m}X \xrightarrow{BP} B\text{Conf}_nX) = B((x_1, \ldots, x_n) \setminus p),$$

where $(x_1, \ldots, x_n)$ is any configuration in $\text{obj}(\text{Conf}_nX)$. One should also recall the definition of homotopy fibre,

$$\text{hofib}_*(f : X \to Y) = \{(x, \gamma) \mid x \in X, \gamma : [0, 1] \to Y, \gamma(0) = *, \gamma(1) = f(x)\},$$

where $* \in Y$ is a base-point. After this theorem is applied, all that would then remain to show would be checking that $B((x_1, \ldots, x_n) \setminus p)$ is homotopically discrete.

If $b : B' \to B$ is an arrow in $\text{Conf}_nX$, then it is the case that $b$ induces a homotopy equivalence

$$B(B \setminus p) \xrightarrow{\simeq} B(B' \setminus p).$$

To achieve this result, show that $b$ induces an isomorphism of categories,

$$B \setminus p \to B' \setminus p.$$

Let $B = (x_1^1, \ldots, x_n^1) \in \text{obj}(\text{Conf}_nX)$ and $B' = (x_1^2, \ldots, x_n^2) \in \text{obj}(\text{Conf}_nX)$ such that there is an arrow $(x_1^2, \ldots, x_n^2) \to (x_1^1, \ldots, x_n^1)$ in $\text{Conf}_nX$. This means that there exists $\sigma \in S_n$ such that for each $i \in \{1, \ldots, n\}$ there is an arrow $\alpha_i : x_i^2 \to x_{\sigma(i)}^1$ in $X_1$.

The category $B \setminus p$ has:

- Objects in $B \setminus p$ are pairs $(c, f)$ with $c = (y_1, \ldots, y_n) \in \text{obj}(\text{Conf}_{n,m}X)$ and $f : (x_1^1, \ldots, x_n^1) \to p(c)$ an arrow in $\text{Conf}_nX$. Recall that $p$ is the identity map on objects, so $p(c) = c = (y_1, \ldots, y_n)$. Such an $f$ exists if and only if there is some $\rho \in S_n$ such that at each $i \in \{1, \ldots, n\}$ there is an arrow $y_i \to x_{\rho(i)}^1$ in $X_1$.

- Arrows in $B' \setminus p$ from $(\vec{g}, f)$ to $(\vec{z}, g)$ are arrows $h : (y_1, \ldots, y_n) \to (z_1, \ldots, z_n)$.

\footnote{The reason for using $\sigma^{-1}$ rather than just $\sigma$ in $x_i^2 \to x_{\sigma^{-1}(i)}^1$ will become clear later.}
in $Conf_{n,m}\mathcal{X}$ such that

\[
\begin{array}{ccc}
(x_1^1, \ldots, x_n^1) & \xrightarrow{f} & (x_1^2, \ldots, x_n^2) \\
p(y_1, \ldots, y_n) & \xrightarrow{p(h)} & p(z_1, \ldots, z_n)
\end{array}
\]

commutes in $Conf_n\mathcal{X}$.

Write $b^*$ for the functor induced by $b : B' \to B$,

\[b^* : (x_1^1, \ldots, x_n^1) \setminus p \to (x_1^2, \ldots, x_n^2) \setminus p.\]

The map $b^*$ needs to be described on both objects and morphisms, followed by showing that it is an isomorphism of categories. On objects, $b^*$ is the map

\[b^*_0 : \text{obj}(B \setminus p) \to \text{obj}(B' \setminus p), (c, f) \mapsto (c, f \circ b).\]

One can check that since $f : B \to p(c)$ and $b : B' \to B$ then $f \circ b : B' \to p(c)$, so $(c, f \circ b)$ is indeed an object in $B' \setminus p$. On morphims $b^*$ effectively acts as the identity map. To see that this is the case, recall that an arrow in $B_n \setminus p$ from $(\vec{y}, f)$ to $(\vec{z}, g)$ is an arrow $h : \vec{y} \to \vec{z}$ in $Conf_{n,m}\mathcal{X}$ so that

\[
\begin{array}{ccc}
B & \xrightarrow{f} & \xrightarrow{g} & B' \\
p(\vec{y}) & \xrightarrow{p(h)} & p(\vec{z})
\end{array}
\]

commutes in $Conf_n\mathcal{X}$. Since it is the case that $b : B' \to B$ is an arrow in $Conf_n\mathcal{X}$, one can see that

\[
\begin{array}{ccc}
B' & \xrightarrow{b} & B \\
\xrightarrow{f \circ b} & & \xleftarrow{g \circ b} \\
p(\vec{y}) & \xrightarrow{p(h)} & p(\vec{z})
\end{array}
\]

is a commutative diagram in $Conf_n\mathcal{X}$. The outer triangle shows that $h : \vec{y} \to \vec{z}$ defines an arrow in $B' \setminus p$.

The aim is to show that this induced map, $b^*$, is an isomorphism of categories. The arrow $b : B' \to B$, being an arrow in $Conf_n\mathcal{X}$, is made up of a re-ordering
of the elements and \( n \) arrows from \( X_1 \). Keeping the notation from above, the map \( b \) will be made up of \( \sigma \in S_n \) and an arrow for each \( i \in \{1, \ldots, n\} \) of the form \( \alpha_i : x_i^2 \to x_{\sigma^{-1}(i)}^1 \). One can now think of \( b \) as applying \( \alpha = (\alpha_1, \ldots, \alpha_n) \) to \( B' \) followed by re-ordering the elements by \( \sigma \),

\[
b : B' \to B,
(x_1^2, \ldots, x_n^2) \mapsto \sigma \circ \alpha(x_1^2, \ldots, x_n^2),
= \sigma(\alpha_1(x_1^2), \ldots, \alpha_n(x_n^2)),
= (\alpha_{\sigma(1)}(x_{\sigma(1)}^2), \ldots, \alpha_{\sigma(n)}(x_{\sigma(n)}^2)),
= (x_{\sigma^{-1}(\sigma(1))}^1, \ldots, x_{\sigma^{-1}(\sigma(n))}^1),
= (x_1^1, \ldots, x_n^1).
\]

The inverse of this map is \( b^{-1} = \alpha^{-1} \circ \sigma^{-1} \), or, if one wishes to write it in the same applying arrows followed by a re-ordering way, \( b^{-1} = \sigma^{-1} \circ \gamma \), where \( \gamma = \sigma(\alpha^{-1}) \). To see this is the case, consider both of these descriptions for \( b^{-1} \) acting on \( B = (x_1^1, \ldots, x_n^1) \) and note that \( \alpha_i : x_i^2 \to x_{\sigma^{-1}(i)}^1 \) implies that both \( \alpha_i^{-1} : x_{\sigma^{-1}(i)}^1 \to x_i^2 \) and \( \alpha_{\sigma(i)}^{-1} : x_i^1 \to x_{\sigma(i)}^2 \). Checking the first case,

\[
b^{-1}(B) = \alpha^{-1} \circ \sigma^{-1}(x_1^1, \ldots, x_n^1),
= (\alpha_1^{-1}, \ldots, \alpha_n^{-1})(x_{\sigma^{-1}(1)}^1, \ldots, x_{\sigma^{-1}(n)}^1),
= (\alpha_1^{-1}(x_{\sigma^{-1}(1)}^1), \ldots, \alpha_n^{-1}(x_{\sigma^{-1}(n)}^1)),
= (x_1^1, \ldots, x_n^2),
\]

and in the second case,

\[
b^{-1}(B) = \sigma^{-1} \circ \gamma(x_1^1, \ldots, x_n^1),
= \sigma^{-1} \circ (\sigma(\alpha^{-1}))(x_1^1, \ldots, x_n^1),
= \sigma^{-1} \circ (\sigma(\alpha_1^{-1}, \ldots, \alpha_n^{-1}))(x_1^1, \ldots, x_n^1),
= \sigma^{-1} \circ (\alpha_{\sigma(1)}^{-1}, \ldots, \alpha_{\sigma(n)}^{-1})(x_{\sigma(1)}^1, \ldots, x_{\sigma(n)}^1),
= \sigma^{-1}(\alpha_{\sigma(1)}^{-1}(x_{\sigma(1)}^1), \ldots, \alpha_{\sigma(n)}^{-1}(x_{\sigma(n)}^1)),
= \sigma^{-1}(x_{\sigma(1)}^2, \ldots, x_{\sigma(n)}^2),
= (x_1^2, \ldots, x_n^1),
= (x_1^1, \ldots, x_n^2),
\]

shows that both descriptions of \( b^{-1} \) produce the same result. As with the induced map \( b^* \) on the comma categories, the map induced by the inverse map
on the comma categories is defined,

$$ (b^{-1})^* : B' \setminus p \to B \setminus p. $$

On objects the induced map is

$$ (b^{-1})^*_0 : \text{obj}(B' \setminus p) \to \text{obj}(B \setminus p), $$

$$ (c, f) \mapsto (c, f \circ b^{-1}), $$

on morphisms, as was the case for the $b^*$ map, $(b^{-1})^*$ effectively acts as the identity.

Checking the composition of the two induced maps, one finds that $(b^{-1})^* = (b^*)^{-1}$. On objects, observe that

$$ (b^{-1})^*_0 \circ b^*_0 : \text{obj}(B \setminus p) \to \text{obj}(B \setminus p), $$

$$ (b^{-1})^*_0 \circ b^*_0(c, f) = (b^{-1})^*_0(c, f \circ b), $$

$$ = (c, f \circ b \circ b^{-1}), $$

$$ = (c, f \circ id), $$

$$ = id(c, f), $$

and,

$$ b^*_0 \circ (b^{-1})^*_0 : \text{obj}(B' \setminus p) \to \text{obj}(B' \setminus p), $$

$$ b^*_0 \circ (b^{-1})^*_0(c', f') = b^*_0(c', f' \circ b^{-1}), $$

$$ = (c', f' \circ b^{-1} \circ b), $$

$$ = (c', f' \circ id), $$

$$ = id(c', f'). $$

Similarly with the map on morphisms, the commutative diagrams for $(b^{-1})^*_1 \circ b^*_1$ and $b^*_1 \circ (b^{-1})^*_1$, respectively, become,
3.3. A COVERING MAP UP TO HOMOTOPY

respectively. In the diagrams above, the inner triangle is the original commutative diagram for a morphism in $B \setminus p$. For the left diagram, the middle triangle is the commutative diagram for the image of the morphism under $b_1^*$, and the outer triangle is the commutative diagram for the image of the original morphism under $(b^{-1})_1^* \circ b_1^*$. Similarly for the right diagram above. Note that in both cases, the outer triangle is equal to the inner triangle giving the result that $(b^{-1})_1^* \circ b_1^* = id$ and $b_1^* \circ (b^{-1})_1^* = id$.

Using this information, it is the case that $(b_1)$ is the identity functor on $B \setminus p$ and $b \circ (b^{-1})$ is the identity functor on $B' \setminus p$. Therefore, the map $b : B' \to B$ induces a homotopy equivalence on the classifying spaces of the comma categories,

$$B(B \setminus p) \xrightarrow{\sim} B(B' \setminus p).$$

This shows that the map $p : Conf_{n,m} X \to Conf_n X$ satisfies the assumption of Theorem 3.3.2. Therefore, the map $BP$ has homotopy fibre $B(B \setminus p)$.

The task now becomes showing that $B(B \setminus p)$ is homotopically discrete.

Claim 3.3.3. Let $(x_1, \ldots, x_n) \in \text{obj}(Conf_n X)$ and let $\overline{S_{n,m}}$ be a set of coset representatives of $S_n/(S_{n-m} \times S_m)$. A skeletal subcategory of $(x_1, \ldots, x_n) \setminus p$ is the category with,

- objects:

$$\{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \sigma \in \overline{S_{n,m}}\};$$

and

- morphisms being the identity morphisms.

Note that if $S$ is the set of objects of this skeletal subcategory, then $BS \simeq S$ with the discrete topology.

Proof of Claim. The process of showing that this is a skeletal subcategory is simply checking that there is exactly one representative for each object in $(x_1, \ldots, x_n) \setminus p$.

This proof will be done in two parts:

1. if $(\overline{y}, f)$ is an object in $(x_1, \ldots, x_n) \setminus p$ then there exists $\sigma \in \overline{S_{n,m}}$ such that $(\overline{y}, f)$ is connected to $((x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \sigma)$ by an arrow; and

2. if $\sigma_1, \sigma_2 \in \overline{S_{n,m}}$ such that $\sigma_1 \neq \sigma_2$ then there is no arrow from $((x_{\sigma_1(1)}, \ldots, x_{\sigma_1(n)}), \sigma_1)$ to $((x_{\sigma_2(1)}, \ldots, x_{\sigma_2(n)}), \sigma_2)$. 


CHAPTER 3. HOMOLOGICAL INJECTIVITY

Beginning with the first part, take \((\vec{y}, f) \in \text{obj}(\langle x_1, \ldots, x_n \rangle \setminus p)\). The aim is to find a \(\sigma \in \widetilde{S}_{n,m}\) such that there is an arrow \((\vec{y}, f) \to (\sigma(\vec{x}), \sigma)\). By the definition of \((x_1, \ldots, x_n) \setminus p\), it is the case that the arrow \(f \in S_{n-m} \times S_m\) and arrows \(\alpha_i \in X_1\) such that,

\[ \alpha_1 : x_1 \to y_{\rho^{-1}(1)}, \ldots, \alpha_n : x_n \to y_{\rho^{-1}(n)}. \]

As before, think of the arrow as the composite \(f = \rho \circ \alpha\). Recall that an arrow from \((\vec{y}, f)\) to \((\vec{z}, f')\) in \(B \setminus p\) is an arrow \(h : \vec{y} \to \vec{z}\) in \(\text{mor}(Conf_{n,m}\mathcal{X})\) such that \(f \circ p(h) = f'\). The \(\alpha_i\) and \(\rho\) above give an arrow in \(Conf_{n,m}\mathcal{X}\), namely \(\rho(\alpha^{-1})\), which in turn gives the arrow in the comma category

\[ ((y_1, \ldots, y_n), f) \to ((x_{\rho(1)}, \ldots, x_{\rho(n)}), \rho). \]

To show that this indeed defines an arrow from \((\vec{y}, f)\) to \((\langle x_{\rho(1)}, \ldots, x_{\rho(n)} \rangle, \rho)\) observe that \(\rho(\alpha^{-1}) : \vec{y} \to \vec{x}\),

\[
\rho(\alpha^{-1})(y_1, \ldots, y_n) = (\alpha_{\rho(1)}^{-1}(y_1), \ldots, \alpha_{\rho(n)}^{-1}(y_n)), \\
= (x_{\rho(1)}(y_1), \ldots, x_{\rho(n)}(y_n)),
\]

and \(p(\rho(\alpha^{-1})) \circ f = p(\rho(\alpha^{-1}) \circ \rho \circ \alpha)\) (substituting in \(f\) and noting that \(p\) is the identity map on arrows). Using work from above (the two different descriptions of \(b^{-1}\)), this equation can be modified by replacing \(\rho \circ \alpha\) with \((\rho^{-1} \circ \rho(\alpha^{-1}))^{-1}\), the maps become,

\[
p(\rho(\alpha^{-1})) \circ f = \rho(\alpha^{-1}) \circ \rho \circ \alpha, \\
= \rho(\alpha^{-1}) \circ (\rho^{-1} \circ \rho(\alpha^{-1}))^{-1}, \\
= \rho(\alpha^{-1}) \circ (\rho(\alpha^{-1}))^{-1} \circ \rho, \\
= \rho,
\]

as required for \(\rho(\alpha^{-1})\) to define an arrow in from \((\vec{y}, f)\) to \((\rho(\vec{x}), \rho)\) (the commutative diagram for arrows in the comma category).

Since \(\widetilde{S}_{n,m}\) contains a coset representative for each coset of \(S_n / (S_{n-m} \times S_m)\), there exists a \(\sigma \in \widetilde{S}_{n,m}\) such that \(\rho \in [\sigma]\), the coset which contains \(\sigma\). Since both \(\rho, \sigma \in [\sigma]\), it is the case that \(\sigma \circ \rho^{-1} \in S_{n-m} \times S_m\). This arrow \(\sigma \circ \rho^{-1}\) is an arrow in \(Conf_{n,m}\mathcal{X}\) which defines an arrow in \(B \setminus p\),

\[ \sigma \circ \rho^{-1} : ((x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \rho) \to ((x_{\rho(1)}, \ldots, x_{\rho(n)}), \sigma). \]
3.4. PROOF OF HOMOLOGICAL INJECTIVITY

Composing these arrows gives an arrow
\[ \sigma \circ \rho^{-1} \circ \rho(\alpha^{-1}) : ((y_1, \ldots, y_n), f) \rightarrow ((x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \sigma), \]
where \( \sigma \in \tilde{S}_{n,m} \). Thus the first part of the claim is true.

To show the second part of the claim, it needs to be shown that if \( \sigma_1, \sigma_2 \in \tilde{S}_{n,m} \) such that \( \sigma_1 \neq \sigma_2 \) then there is no arrow \( (\sigma_1(\vec{x}), \sigma_1) \rightarrow (\sigma_2(\vec{x}), \sigma_2) \). Suppose such an arrow did exist, the following will show that this causes a contradiction. Assuming the existence of such an arrow, there would exist an arrow in \( \text{Conf}_{n,m} \mathcal{X} \) of the form,
\[ (x_{\sigma_1(1)}, \ldots, x_{\sigma_1(n)}) \rightarrow (x_{\sigma_2(1)}, \ldots, x_{\sigma_2(n)}). \]
The arrows in \( \text{Conf}_{n,m} \mathcal{X} \) are formed from a re-ordering \( \rho \in S_{n-m} \times S_m \) and \( n \) arrows \( \alpha_i : x_{\sigma_1(i)} \rightarrow x_{\rho \sigma_2(i)} \) in \( X_1 \). Since \( \sigma_2(\vec{x}) \) is simply a re-ordering of \( \sigma_1(\vec{x}) \), and there are no arrows \( x_i \rightarrow x_j \) when \( i \neq j \), the \( \alpha_i \) maps must satisfy \( s(\alpha_i) = t(\alpha_i) \), equivalently \( x_{\sigma_1(i)} = x_{\rho \sigma_2(i)} \). Therefore, it is the case that \( \rho \circ \sigma_1 = \sigma_2 \). However, recalling that \( \rho \in S_{n-m} \times S_m \), this implies that \( \sigma_1 \) and \( \sigma_2 \) are both in the same coset of \( S_n/(S_{n-m} \times S_m) \). This is a contradiction as \( \tilde{S}_{n,m} \) was defined to contain exactly one element of each coset. Therefore there can be no arrow \( (\sigma_1(\vec{x}), \sigma_1) \rightarrow (\sigma_2(\vec{x}), \sigma_2) \) for \( \sigma_1, \sigma_2 \in \tilde{S}_{n,m} \) with \( \sigma_1 \neq \sigma_2 \).

This concludes the proof that the category with objects \( \{ (\sigma(\vec{x}), \sigma) \mid \sigma \in \tilde{S}_{n,m} \} \) and morphisms being only the identity arrows, is a skeletal subcategory of \( B \setminus p \).

This skeletal subcategory of \( B \setminus p \) is discrete, therefore its classifying space is homotopically discrete. Using the homotopy equivalence of this skeletal subcategory to the original category \( B \setminus p \), one discovers that the original category \( B \setminus p \) is also homotopically discrete. Putting this all together implies that the map \( Bp : B\text{Conf}_{n,m} \mathcal{X} \rightarrow B\text{Conf}_n \mathcal{X} \) is a covering space up to homotopy.

3.4 Proof of Homological Injectivity

This section shows homological injectivity following the same technique as used by Randal-Williams in Section 7 of [RW11], see also [McD75]. Section 3.1 described how to pick \( \varepsilon \in \partial X_0 \) and defined the stabilisation map \( s_\varepsilon : \text{Conf}_n X \rightarrow \text{Conf}_{n+1} X \).

This section will use a different description for the configuration orbispace with \( m \) grouped points.

**Proposition 3.4.1.** The configuration orbispace \( \text{Conf}_{n,m} \mathcal{X} \) as defined in the
previous section (Section 3.3), can alternatively be thought of as the category with

- object space,

\[ \{(x_1, \ldots, x_n) \in \text{obj}(\text{Conf}_n X) \mid x_1, \ldots, x_n \subseteq \{x_1, \ldots, x_n\}, |S| = m \}; \]

- the morphism space is simply a morphism from \( \text{Conf}_n X \) which ‘preserves’ the choice of \( S \). To be explicit, a morphism from \((x_1, \ldots, x_n), S\) to \((y_1, \ldots, y_n), S'\) is a choice of \( n \) arrows \( \alpha_i \in X_1 \) and a \( \sigma \in S_n \) such that for \( \alpha_i : x_i \to y_{\sigma(i)}, x_i \in S \) if and only if \( y_{\sigma(i)} \in S' \).

It is important to note that the \( \sigma \) in the description of the morphisms above is a bit of a red-herring. The only reason it is there is so that the indices on the \( y_i \) and \( z_i \) match up properly. By the construction of the object space, there is at most one such \( \sigma \) for each pair so that the morphism space is non-empty.

Another thing to realise here is that in the definition of the morphism space, the preserving of the choice of \( S \) gives an implicit grouping of the arrows \( \{\alpha_1, \ldots, \alpha_n\} \) into a group of size \( m \) and a group of size \( n - m \).

**Proof of Proposition.** Let \( X \) be an orbifold. Let \( Y \) be the configuration orbispace with \( m \) grouped points as defined in Section 3.3. That is, \( Y \) is the orbispace with

- object space

\[ \text{obj}(\text{Conf}_n X); \]

- the arrows from \( x^1 \) to \( x^2 \) being a choice of \( \sigma \in S_m \times S_{n-m} \) and arrows \( \gamma_1, \ldots, \gamma_n \in X_1 \) such that

\[ \gamma_i : x^1_i \to x^2_{\sigma(i)}, \quad i \in \{1, \ldots, n\}. \]

Let \( Z \) be the orbispace described in the proposition.

The candidate Morita equivalence map \( \varphi : Y \to Z \), on objects is the map

\[ \varphi_0 : (x_1, \ldots, x_n) \mapsto (\{x_1, \ldots, x_n\}, \{x_1, \ldots, x_m\}), \]

and on morphisms is the map

\[ \varphi_1 : (\gamma_1 : x_1 \to y_1, \ldots, \gamma_n : x_n \to y_n) \mapsto \{\gamma_1 : x_1 \to y_1, \ldots, \gamma_n : x_n \to y_n\}. \]
This map is well defined since on the right-hand-side it is the first $m$ elements in both cases, $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$, which make up the subset of size $m$. So the maps $\gamma_i$ do indeed ‘preserve’ the choice of $S$.

Recalling the definition of Morita equivalence (Definition 2.1.14), the conditions that $\varphi$ must satisfy are:

1. the map

$$t \circ \text{proj}_1 : Z_1 \times \varphi Y_0 \to Z_0,$$

$$(g, y) \mapsto t(g),$$

is a surjective local homeomorphism; and

2. the square

$$\begin{array}{ccc}
Y_1 & \xrightarrow{\varphi} & Z_1 \\
\downarrow^{(s,t)} & & \downarrow^{(s,t)} \\
Y_0 \times Y_0 & \xrightarrow{\varphi \times \varphi} & Z_0 \times Z_0
\end{array}$$

is a fibred product.

Begin with the first condition. It isn’t too hard to see that the map $t \circ \text{proj}_1$ is surjective. Take any point in $Z_0$, say $(\underline{z}, S) \in Z_0$. Since $\underline{z}$ is just a set, it’s elements can be re-ordered without changing the object. Re-write the element as $(\{z_1, \ldots, z_n\}, S)$ so that $z_1, \ldots, z_m \in S$ (that is, the $m$ elements of $S$ are written first). Then the point $(\text{id}(\underline{z}, S), (z_1, \ldots, z_n))$ is in the pre-image of $(\underline{z}, S)$ under the map $t \circ \text{proj}_1$.

As in the previous Morita equivalence proof (see Proposition 2.3.6), using that $X$ is an orbifold reduces the local homeomorphism condition to checking that $\varphi_0$ is a local homeomorphism. Recall the map $\varphi_0$ is the map that forgets the ordering on $\vec{x}$ but remembers the first $m$ points in the set $S$,

$$\varphi_0: (x_1, \ldots, x_n) \mapsto (\{x_1, \ldots, x_n\}, \{x_1, \ldots, x_m\}).$$

In a similar way to the proof of Proposition 2.3.6, the Hausdorff property of $X_0$ allows one to choose open neighbourhoods $U_i \subseteq X_0$ such that $x_i \in U_i$ and if $i \neq j$ then $U_i \cap U_j = \emptyset$. An open neighbourhood of $\vec{x}$ is $U = U_1 \times \cdots \times U_n \subseteq Y_0$. The image of $U$ under $\varphi_0$ will be a subset of

$$\left((U_1^{m})^n / S_n\right) \times \left((U_1^{m})^m / S_m\right).$$
CHAPTER 3. HOMOLOGICAL INJECTIVITY

The specific subset is,

\[
\begin{cases}
(x, g) & \{x_1, \ldots, x_n\} \in (\cup_{i=1}^m U_i)^n / S_n, \\
        & \{s_1, \ldots, s_m\} \in (\cup_{i=1}^m U_i)^m / S_m, \\
        & \text{each } s_i \text{ equals } x_j \text{ for some } j, \\
        & \text{each } U_i \text{ contains exactly one } x_j, \\
        & \text{if } i \neq j \text{ then } s_i \neq s_j
\end{cases}
\]

Note that the third, fourth and fifth conditions also enforce that if \(i \in \{1, \ldots, m\}\) then \(U_i\) contains exactly one \(s_j\). From this observation, there is exactly one choice for the \(s_j\) which lies in \(U_i\) after the set \(x\) has been picked. So the subset above is homeomorphic to the set

\[
\left\{ x \in (\cup_{i=1}^m U_i)^n / S_n \mid \text{each } U_i \text{ contains exactly one } x_j \right\}
\]

Using the fact that the \(U_i\) are disjoint, this yields the result that

\[
\left\{ x \in (\cup_{i=1}^m U_i)^n / S_n \mid \text{each } U_i \text{ contains exactly one } x_j \right\} 
\approx U_1 \times \cdots \times U_n,
\]

showing that \(\varphi_0\) is a local homeomorphism.

The second condition of Morita equivalence requires that the square

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{\varphi} & Z_1 \\
\downarrow_{(s,t)} & & \downarrow_{(s,t)} \\
Y_0 \times Y_0 & \xrightarrow{\varphi \times \varphi} & Z_0 \times Z_0
\end{array}
\]

is a fibred product. That is, if there exists a space \(U\) and maps \(q_1 : U \to Y_0 \times Y_0\), \(q_2 : U \to Z_1\) such that the square

\[
\begin{array}{ccc}
U & \xrightarrow{q_2} & Z_1 \\
\downarrow_{q_1} & & \downarrow_{(s,t)} \\
Y_0 \times Y_0 & \xrightarrow{\varphi \times \varphi} & Z_0 \times Z_0
\end{array}
\]

commutes, then there exists an arrow \(u : U \to Y_1\) such that \((s,t) \circ u = q_1\) and \(\varphi \circ u = q_2\).

Take a point \((z, S), (z', S') \in Z_0 \times Z_0\) which is in the image of both \((s, t) \circ q_2\) and \((\varphi \times \varphi) \circ q_1\). As noted above, since \(z\) and \(z'\) are just sets, it is possible to re-label their elements so that the first \(m\) points in each are the ones that appear in \(S\) and \(S'\) respectively. That is, \(z = \{z_1, \ldots, z_n\}, S\) and \(z', S' = \{z'_1, \ldots, z'_n\}, S'\).
3.4. PROOF OF HOMOLOGICAL INJECTIVITY

Let \( \{z'_1, \ldots, z'_n\}, S' \) such that if \( i \in \{1, \ldots, m\} \) then \( z_i \in S \) and \( z'_i \in S' \).

The pre-image of this point under the map \((s,t)\) is made up of sets of \( n \) maps

\[
\gamma_1 : z_1 \to z'_{\sigma(1)}, \ldots, \gamma_n : z_n \to z'_{\sigma(n)}.
\]

To add some clarity, write \( T \) along with the set, \( T \) is the set of \( \gamma_i \) with \( s(\gamma_i) \in S \) and \( t(\gamma_i) \in S' \) (by the way that the \( \gamma_i \) labelling was chosen, this should simply be \( T = \{\gamma_1, \ldots, \gamma_m\} \)). Since the elements of \( \bar{z} \) and \( \bar{z}' \) were re-labelled so that \( S \) and \( S' \) are the first \( m \) elements of each, the allowed permutations for this to be an arrow in \( Z_1 \) are \( \sigma \in S_m \times S_{n-m} \). Note that there is exactly one such \( \sigma \in S_m \times S_{n-m} \) such that these \( \gamma_i \) exist. There is at least one due to the assumption that the object in \( Z_0 \times Z_0 \) was assumed to be the image of something in \( Z_1 \). There is at most one because the existence of two would imply an arrow \( z_i \to z_j \) for \( i \neq j \).

The pre-image of the point \( ((z,S),(z',S')) \) under the map \( \varphi \times \varphi \) is the set

\[
\{(z_{\rho(1)}, \ldots, z_{\rho(n)}), (z'_{\rho'(1)}, \ldots, z'_{\rho'(n)}) \mid \rho, \rho' \in S_m \times S_{n-m}\},
\]

which is all possible ordering of the \( z_i \) and \( z'_i \) so that the elements of \( S \) and \( S' \) appear in the first \( m \) positions.

To find the structure of \( U \), one needs to pull these pre-images back under the maps \( q_1 \) and \( q_2 \). For the diagram to commute, an element \( x \in U \) must satisfy:

\[
q_1(x) = ((z_{\rho(1)}, \ldots, z_{\rho(n)}), (z'_{\rho'(1)}, \ldots, z'_{\rho'(n)})); \quad \text{and}
\]

\[
q_2(x) = ((\gamma_1 : z_1 \to z'_{\sigma(1)}, \ldots, \gamma_n : z_n \to z'_{\sigma(n)}), T),
\]

for some \( \gamma_1, \ldots, \gamma_n \in X_1 \) and \( \rho, \rho', \sigma \in S_m \times S_{n-m} \). The task is to find an element \( \bar{\alpha} \in Y_1 \) such that setting \( u(x) = \bar{\alpha} \) defines a map which satisfies \( (s,t) \circ u = q_1 \) and \( \varphi \circ u = q_2 \).

Recall that there is an arrow \( \bar{y} \to \bar{y}' \) in \( Y_1 \) if there exists \( \nu \in S_m \times S_{n-m} \) and arrows \( \alpha_i : y_i \to y'_{\nu(i)} \). Define

\[
u := \sigma \circ \rho \circ \rho'^{-1} \quad \text{and the arrows are} \quad \gamma_{\rho(i)} : z_{\rho(i)} \to z'_{\nu_{\rho(i)}}.
\]

Note that \( \nu \in S_m \times S_{n-m} \) since all of \( \sigma, \rho \) and \( \rho' \) are as well. Checking the \((s,t)\) maps on \( u(x) \),

\[
(s,t)((\gamma_{\rho(1)}, \ldots, \gamma_{\rho(n)}), \nu) = ((z_{\rho(1)}, \ldots, z_{\rho(n)}), (z'_{\rho'(1)}, \ldots, z'_{\rho'(n)})�),
\]
since,
\[\gamma_{\rho(i)} : z_{\rho(i)} \mapsto z_{\rho'(i)}',\]
\[= \gamma_{\rho(i)} : z_{\rho(i)} \mapsto z_{\sigma(\rho(i))};\]
as required. Checking the \(\varphi\) map on \(u(x),\)
\[\varphi((\gamma_{\rho(1)}, \ldots, \gamma_{\rho(n)}), v) = (\{\gamma_{\rho(1)}, \ldots, \gamma_{\rho(n)}\}, T'),\]
where \(T' = \{\gamma_{\rho(1)}, \ldots, \gamma_{\rho(m)}\}\). Note that \(T' = \{\gamma_{\rho(1)}, \ldots, \gamma_{\rho(m)}\} = \{\gamma_1, \ldots, \gamma_m\} = T,\) since \(\rho \in S_m \times S_{n-m}\). Every \(\gamma_i\) appears, and so
\[\varphi((\gamma_{\rho(1)}, \ldots, \gamma_{\rho(n)})) = (\{\gamma_1, \ldots, \gamma_n\}, T),\]
as required.
Therefore \((s,t) \circ u = q_1\) and \(\varphi \circ u = q_2\) showing that the diagram is indeed a fibred product.

The previous section showed that the map \(BConf_{n,m} X \to BConf_n X\) is a covering space up to homotopy. Since \(Bp\) is a covering, the corresponding transfer map exists, allowing the definition of a map,
\[t_{n,m} : H_*(Conf_n X; \mathbb{Z}) \to H_*(Conf_m X; \mathbb{Z}),\]
by \(t_{n,m} = \pi_* \circ p'\) where,
\[p' : H_*(Conf_n X; \mathbb{Z}) \to H_*(Conf_{n,m} X; \mathbb{Z})\]
is the transfer map, and
\[\pi_* : H_*(Conf_{n,m} X; \mathbb{Z}) \to H_*(Conf_m X; \mathbb{Z})\]
is the map on homology induced by forgetting the \(n - m\) group of points in \(Conf_{n,m} X\). For ease of notation, write \(t_n := t_{n,n-1}\) and write \(i_n\) for the map induced by \(s_e,\)
\[i_n : H_*(Conf_n X; \mathbb{Z}) \to H_*(Conf_{n+1} X; \mathbb{Z}).\]

An important ingredient of the proof of homological injectivity is the following lemma ([Dol62] Lemma 2).
3.4. PROOF OF HOMOLOGICAL INJECTIVITY

Lemma 3.4.2. Let

\[ 0 \xrightarrow{\sigma_0} B_0 \xrightarrow{\sigma_1} B_1 \xrightarrow{\sigma_2} \ldots \xrightarrow{\sigma_m} B_m \]

be morphisms in some abelian category \( \mathfrak{A} \). Assume there exist morphisms

\[ \tau_{k,n} : B_n \to B_k, \quad \text{for } k \leq n \leq m \]

such that

\[ \tau_{k,n} \sigma_n \equiv \tau_{k,n-1} \mod(\text{im}(\sigma_k)) \]

and

\[ \tau_{k,k} = \text{Id}. \]

Then the morphism

\[ T_n : B_n \to \bigoplus_{k=0}^n B_k/\text{im}(\sigma_k), \]

with components

\[ B_n \xrightarrow{t_{k,n}} B_k \xrightarrow{\text{proj}} B_k/\text{im}(\sigma_k) = \text{coker}(\sigma_k) \]

is an isomorphism, and \( \sigma_n \) has a left inverse, for \( n \in \{0, \ldots, m\} \).

The above lemma can be applied once the following claim is shown, this is a common technique from the manifold setting.

Claim 3.4.3. The maps \( i_n \) and \( t_{n,m} \) defined above satisfy the identity,

\[ t_{n,m} \circ i_{n-1} = i_{m-1} \circ i_{n-1,m-1} + t_{n-1,m}. \quad (3.1) \]

Before working on the proof of this claim, one should consider an alternate description of the transfer map \( p^! \), which, while not due to [Tra13], it is explained well therein. In Section 3.2, the transfer map was defined only on the homology of spaces,

\[ p^! : H_* Y \to H_* X. \]

An alternate way to define the transfer map is by starting with a map on the level of spaces and using symmetric products due to Dold-Thom in [DT85].
**Definition 3.4.4.** The $n$th symmetric product of a space $X$ is the space

$$Sym^n X := X^n / S_n.$$ Given a base-point $* \in X$, one can define a map

$$Sym^n X \to Sym^{n+1} X,
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, *).$$

Taking the limit as $n$ goes to infinity gives the infinite symmetric product on $X$ which receives injective maps from every $Sym^n X$,

$$Sym^n X \to Sym^\infty X,
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, *, *, \ldots).$$

The Dold-Thom Theorem [DT58] says that:

**Theorem 3.4.5.** Let $X$ be a connected CW-complex with base-point. If $i \in \mathbb{Z}_{\geq 0}$ then

$$H_i X \cong \pi_i(Sym^\infty X).$$

Given an $n$-fold covering $p : X \to Y$, there is a map,

$$\tau : Y \to Sym^n X,
y \mapsto p^{-1}(y).$$

Using a map,

$$Sym^\infty (Sym^n X) \to Sym^\infty X,$$

$$((x^1_1, \ldots, x^1_n), (x^2_1, \ldots, x^2_n), \ldots) \mapsto (x^1_1, \ldots, x^1_n, x^2_1, \ldots, x^2_n, \ldots),$$

the map $\tau$ induces a map

$$Sym^\infty Y \to Sym^\infty (Sym^n X) \to Sym^\infty X.$$

This in turn induces a map on the homotopy groups of these spaces,

$$\tau_* : \pi_* (Sym^\infty Y) \to \pi_* (Sym^\infty X),$$
which, using the Dold-Thom Theorem, is the map

$$\tau_* : H_* Y \to H_* X.$$ 

The map $\tau_*$ is the same transfer map $p^!$ as defined previously. Using this alternate definition of the transfer map, it is possible to prove the claim.

Note here that this proof is substantially more verbose than the usual manifold case. While the general structure closely follows classical proofs, the largest difference is due to the requirement of working with classifying spaces. Thus, rather than simply investigating a single configuration in $M$, the proof requires looking at a chain of configurations in $X$, connected by arrows. Another subtle part of this proof is the requirement to define $f_S$, which needs to take into account arrows, something which does not appear in the standard manifold case.

**Proof of Claim.** The aim is to show that, on homology, the equivalence

$$t_{n,m} \circ i_{n-1} = i_{m-1} \circ t_{n-1,m-1} + t_{n-1,m}$$

holds. The proof will show that the behaviour of the underlying maps on spaces, $BConf_{n-1} X \to Sym^\infty BConf_m X$, give the result. Once the underlying maps are shown to be homotopic, the induced maps, $H_*(Conf_{n-1} X) \to H_*(Conf_m X)$, must be equal.

The underlying map of $t_{n,m} \circ i_{n-1}$ is the composition,

$$BConf_{n-1} X \xrightarrow{s_{i_{n-1}}} BConf_{n-1} X \xrightarrow{trf} Sym^{(n)}(BConf_{n,m} X) \xrightarrow{forget} Sym^{(n)}(BConf_m X). \quad (3.2)$$

Here, $s_{i_{n-1}}$ is the map adding a point to the configuration, $trf$ is the underlying map to the transfer map, that is the set of all possible liftings under the map $p$, and $forget$ is the map which forgets all the points in the $n-m$ grouping of the configuration. The underlying maps of the right-hand side of the equation above have a similar description. For the $i_{m-1} \circ t_{n-1,m-1}$ part, the underlying map is the composition,

$$BConf_{n-1} X \xrightarrow{trf} Sym^{(n-1)}(BConf_{n-1,m-1} X) \xrightarrow{forget} Sym^{(n-1)}(BConf_{n-1,m-1} X) \xrightarrow{s_{i_{m-1}}} Sym^{(n-1)}(BConf_m X), \quad (3.3)$$

and for the $t_{n-1,m}$ part, the underlying map is the composition,

$$BConf_{n-1} X \xrightarrow{trf} Sym^{(n-1)}(BConf_{n-1,m} X) \xrightarrow{forget} Sym^{(n-1)}(BConf_{n-1,m} X). \quad (3.4)$$
The proof of (\ref{equation-proof-homological-injectivity}) becomes showing that starting with any \( x \in BConf_{n-1}\mathcal{X} \), then the set obtained on the right-hand side of (\ref{equation-proof-homological-injectivity}) is the union of the right-hand sides of both (\ref{equation-proof-homological-injectivity}) and (\ref{equation-proof-homological-injectivity}).

All three of these maps on homology have domain \( BConf_{n-1}\mathcal{X} \) and codomain being some symmetric product of \( BConf_m\mathcal{X} \). Intuitively, the original equation is thought of as saying that adding a point followed by forgetting down to \( m \) points is the same as either: forgetting down to \( m-1 \) points followed by adding a point; or just forgetting down to \( m \) points.

Write \( \alpha = \text{forget} \circ \text{trf} \circ s_e \) for the process in Equation (\ref{equation-proof-homological-injectivity}), \( \beta_1 = s_e \circ \text{forget} \circ \text{trf} \) for the process in Equation (\ref{equation-proof-homological-injectivity}) and \( \beta_2 = \text{forget} \circ \text{trf} \) for the process in Equation (\ref{equation-proof-homological-injectivity}). The co-domain of \( \alpha \) is \( \text{Sym}(\binom{n}{m})(BConf_m\mathcal{X}) \), the co-domain of \( \beta_1 \) is \( \text{Sym}(\binom{n-1}{m})(BConf_m\mathcal{X}) \) and the co-domain of \( \beta_2 \) is \( \text{Sym}(\binom{n-1}{m})(BConf_m\mathcal{X}) \).

There is a map on the symmetric products,

\[
\gamma : \text{Sym}(\binom{n-1}{m})(BConf_m\mathcal{X}) \times \text{Sym}(\binom{n-1}{m})(BConf_m\mathcal{X}) \to \text{Sym}(\binom{n}{m})(BConf_m\mathcal{X}),
\]

\[
((a_1, \ldots, a_{\binom{n-1}{m}}), (b_1, \ldots, b_{\binom{n-1}{m}})) \mapsto (a_1, \ldots, a_{\binom{n-1}{m}}, b_1, \ldots, b_{\binom{n-1}{m}}),
\]

this is well defined by recalling the identity \( \binom{n-1}{m} + \binom{n-1}{m} = \binom{n}{m} \).

The aim is to show that if \( ((\bar{x}^1 \leftarrow \cdots \leftarrow \bar{x}^i), t) \in BConf_{n-1}\mathcal{X} \) then

\[
\alpha((\bar{x}^1 \leftarrow \cdots \leftarrow \bar{x}^i), t) \simeq \gamma(\beta_1((\bar{x}^1 \leftarrow \cdots \leftarrow \bar{x}^i), t), \beta_2((\bar{x}^1 \leftarrow \cdots \leftarrow \bar{x}^i), t)).
\]

To see this is the case, consider the images of \( \alpha, \beta_1 \) and \( \beta_2 \). Starting with the point \( ((\bar{x}^1 \leftarrow \cdots \leftarrow \bar{x}^i), t) \in BConf_{n-1}\mathcal{X} \), apply piece by piece the maps which make up \( \alpha = \text{forget} \circ \text{trf} \circ s_e \). The original point in \( BConf_{n-1}\mathcal{X} \) is

\[
((x_1^1, \ldots, x_{n-1}^1) \leftarrow \cdots \leftarrow (x_1^i, \ldots, x_{n-1}^i), t).
\]

Recalling the map \( k : \mathcal{X} \to \mathcal{X} \) defined in Section (\ref{subsection-proof-homological-injectivity}), applying \( s_e \) to this point gives the new point:

\[
((k(x_1^1), \ldots, k(x_{n-1}^1), k(\varepsilon)) \leftarrow \cdots \leftarrow (k(x_1^i), \ldots, k(x_{n-1}^i), k(\varepsilon)), t).
\]

It is important to note what the fibre over this point looks like in \( BConf_{n,m}\mathcal{X} \).

Recall that the definition of configurations with grouped points in this section relied on choosing a configuration \( \{x_1, \ldots, x_n\} \) along with a subset \( S \subseteq \{x_1, \ldots, x_n\} \), \( |S| = m \). It was noted earlier that since the definition of arrows required that they preserve the set \( S \), these groupings implicitly define a grouping on the space of arrows. Using this, observe that the fibre over an arrow \( g \in \text{mor}(Conf_{n}\mathcal{X}) \) will be \( \binom{n}{m} \) copies of \( g \), each one paired with one of the

\[3\text{Hint: } s_e(x_1, \ldots, x_n) = (k(x_1), \ldots, k(x_n), k(\varepsilon)) \]
possible subsets $S \subseteq g$ of size $m$.

Therefore, a point over an arrow in $Conf_n X$ is precisely the arrow again along with a choice of $S$. When there is a chain of arrows in $Conf_n X$ and one investigates the fibre over that chain, the choice of an $S$ at an endpoint will completely determine the choice of $S$ at any point in the chain. It is also the case that if the identity arrow appears in the fibre, then the original arrow it was lifted from in the chain of $Conf_n X$ must have been the identity. These observations are important as it shows that if one looks at the fibre over a point in an $n$-dimensional non-degenerate simplex of $Conf_n X$, then it will be made up of $\binom{n}{m}$ points, each one in an $n$-dimensional non-degenerate simplex of $BConf_{n,m} X$.

Using this knowledge, applying the transfer map gives the following object in $Sym^{\binom{n}{m}}(BConf_{n,m} X)$,

$$
\left\{ \left( (k(x_1), \ldots, k(x_{n-1}), k(\varepsilon)), \ldots \right) \right| \begin{array}{l}
S \subseteq \{k(x_1), \ldots, k(x_{n-1}), k(\varepsilon)\}, \\
|S| = m
\end{array}
\right\}.$$

This uses the fact that the fibre over a non-degenerate $n$-simplex is made up of non-degenerate $n$-simplices as it allows one to simply copy the $t \in \Delta^n$ across when applying the underlying map of the transfer.

Before looking at what the $forget$ map does to this space, there is one more thing that is needed. Let,

- $(y_1, \ldots, y_n) \in Conf_n X$; and
- $S$ be a set of $m$ distinct points in $X_0$ such that each point in $S$ is connected to a distinct $y_i$ by an arrow in $X_1$ (note that this also ensures that no two elements of $S$ are connected by an arrow).

Then define,

$$f_S(y_1, \ldots, y_n) = (\hat{y}_1, \ldots, \hat{y}_n),$$

where $y_i$ appears in the right hand tuple if and only if there exists an arrow from $y_i$ to an element of $S$. The right-hand-side of this equation is an element of $Conf_m X$.

Applying the $forget$ map gives the final element of $Sym^{\binom{n}{m}}(BConf_{m} X)$,

$$
\left\{ \left( f_S(k(x_1), \ldots, k(x_{n-1}), k(\varepsilon)), \ldots \right) \right| \begin{array}{l}
S \subseteq \{k(x_1), \ldots, k(x_{n-1}), k(\varepsilon)\}, \\
|S| = m
\end{array}
\right\}.$$

This is the image of \((\vec{x}^1 \leftarrow \cdots \leftarrow \vec{x}^l, t)\) under the map \(\alpha\), the next step is to do the same process with the \(\beta_1\) and \(\beta_2\) maps then compare the results.

Repeating this process with \(\beta_1 = s_\varepsilon \circ \text{forget} \circ \text{trf}\), start with the same point in \(BConf_{n-1}\mathcal{X}\),

\[
((x^1_1, \ldots, x^1_{n-1}) \leftarrow \cdots \leftarrow (x^l_1, \ldots, x^l_{n-1}), t).
\]

First, apply the underlying map to the transfer map, \(H_* (Conf_{n-1}\mathcal{X}) \rightarrow H_* (Conf_{n-1,m-1}\mathcal{X})\),

\[
\left\{ \begin{array}{ll}
(x^1_1, \ldots, x^1_{n-1}) \leftarrow & S \subseteq \{x^1_1, \ldots, x^1_{n-1}\}, \\
\cdots & |S| = m - 1 \\
(x^l_1, \ldots, x^l_{n-1}) & 
\end{array} \right\}.
\]

Forgetting down to \(m - 1\) points gives the set,

\[
\left\{ \begin{array}{ll}
fs(x^1_1, \ldots, x^1_{n-1}) \leftarrow & S \subseteq \{x^1_1, \ldots, x^1_{n-1}\}, \\
\cdots & |S| = m - 1 \\
fs(x^l_1, \ldots, x^l_{n-1}) & 
\end{array} \right\}.
\]

Adding the point with \(s_\varepsilon\) gives,

\[
\left\{ \begin{array}{ll}
fs(k(x^1_1), \ldots, k(x^1_{n-1})), k(\varepsilon) \leftarrow & S \subseteq \{k(x^1_1), \ldots, k(x^1_{n-1})\}, \\
\cdots & |S| = m - 1 \\
fs(k(x^l_1), \ldots, k(x^l_{n-1})), k(\varepsilon) & 
\end{array} \right\},
\]

which is the image of \((x^1 \leftarrow \cdots \leftarrow x^l), t)\) under \(\beta_1\).

Similarly with \(\beta_2 = \text{forget} \circ \text{trf}\) applied to the same point in \(BConf_{n-1}\mathcal{X}\),

\[
((x^1_1, \ldots, x^1_{n-1}) \leftarrow \cdots \leftarrow (x^l_1, \ldots, x^l_{n-1}), t),
\]

the underlying map to the transfer, \(H_* (Conf_{n-1}\mathcal{X}) \rightarrow H_* (Conf_{n-1,m}\mathcal{X})\), gives,

\[
\left\{ \begin{array}{ll}
(x^1_1, \ldots, x^1_{n-1}) \leftarrow & S \subseteq \{x^1_1, \ldots, x^1_{n-1}\}, \\
\cdots & |S| = m \\
(x^l_1, \ldots, x^l_{n-1}) & 
\end{array} \right\}.
\]

Forgetting down to the \(m\) grouped points gives,

\[
\left\{ \begin{array}{ll}
fs(x^1_1, \ldots, x^1_{n-1}) \leftarrow & S \subseteq \{x^1_1, \ldots, x^1_{n-1}\}, \\
\cdots & |S| = m \\
fs(x^l_1, \ldots, x^l_{n-1}) & 
\end{array} \right\}.
\]
3.4. PROOF OF HOMOLOGICAL INJECTIVITY

There is then a homotopy to the objects,

\[
\left\{ \left( \frac{f_S(k(x^1)), \ldots, k(x^1_{n-1})}{f_S(k(x^1), \ldots, k(x^1_{n-1}))} \right) \mid S \subseteq \{k(x^1), \ldots, k(x^1_{n-1})\}, |S| = m \right\},
\]

which is homotopic to the image of \((x^1 \leftarrow \cdots \leftarrow x^1), t\) under \(\beta_2\).

Comparing these three images of \((x^1 \leftarrow \cdots \leftarrow x^1), t\), one has the image under \(\alpha\):

\[
\left\{ \left( \frac{f_S(k(x^1), \ldots, k(x^1_{n-1}), k(\varepsilon))}{f_S(k(x^1), \ldots, k(x^1_{n-1}), k(\varepsilon))} \right) \mid S \subseteq \{k(x^1), \ldots, k(x^1_{n-1}), k(\varepsilon)\}, |S| = m \right\},
\]

the image under \(\beta_1\):

\[
\left\{ \left( \frac{f_S(k(x^1), \ldots, k(x^1_{n-1}), k(\varepsilon))}{f_S(k(x^1), \ldots, k(x^1_{n-1}), k(\varepsilon))} \right) \mid S' \subseteq \{k(x^1), \ldots, k(x^1_{n-1})\}, |S'| = m - 1 \right\},
\]

and the image under \(\beta_2\):

\[
\left\{ \left( \frac{f_S(k(x^1), \ldots, k(x^1_{n-1}))}{f_S(k(x^1), \ldots, k(x^1_{n-1}))} \right) \mid S' \subseteq \{k(x^1), \ldots, k(x^1_{n-1})\}, |S'| = m \right\}.
\]

Starting with an element in Equation (3.5), consider the components of the set \(S\). The set \(S\) is a choice of \(m\) elements from the set \(\{k(x^1), \ldots, k(x^1_{n-1}), k(\varepsilon)\}\). There are two cases:

1. If \(k(\varepsilon) \in S\), then \(k(\varepsilon)\) survives the application of \(f_S\) and appears in the final object. The set \(S\) is then made up of \(k(\varepsilon)\), along with a choice of \(m - 1\) elements from the set \(\{k(x^1), \ldots, k(x^1_{n-1})\}\). This is precisely the set described in Equation (3.6) with the set \(S' = S \setminus k(\varepsilon)\).

2. If \(k(\varepsilon) \notin S\), then \(k(\varepsilon)\) is killed off by the application of \(f_S\) and does not appear in the final object. Since the set \(S\) does not contain \(k(\varepsilon)\), it simply consists of a choice of \(m\) elements from the set \(\{k(x^1), \ldots, k(x^1_{n-1})\}\). This is the set described in Equation (3.7) with the set \(S' = S\).
This shows that for \((x^1 \leftarrow \cdots \leftarrow x^l, t) \in B\text{Conf}_{n-1}\mathcal{X}\),
\[
\alpha(x^1 \leftarrow \cdots \leftarrow x^l, t) \simeq \gamma(\beta_1(x^1 \leftarrow \cdots \leftarrow x^l, t), \beta_2(x^1 \leftarrow \cdots \leftarrow x^l, t)).
\]
The maps on the underlying spaces being homotopic implies that the maps on homology are equal, giving the result that
\[
t_{n,m} \circ i_{n-1} = i_{m-1} \circ t_{n-1,m-1} + t_{n-1,m}.
\]
\[\square\]

It is the case that \(t_{m+1} \circ t_{m+2} \circ \cdots \circ t_n = (n-m)! \cdot t_{n,m}\). This is because there are \((n-m)!\) ways to order the \(n - m\) points which are getting removed from the configuration. Using this one can obtain the relation
\[
t_{m,k} \circ t_{n,m} = \left(\frac{n-k}{n-m}\right) \cdot t_{n,k}.
\]
To why this relation is true, use the \(t_{m+1} \circ t_{m+2} \circ \cdots \circ t_n = (n-m)! \cdot t_{n,m}\),
\[
LHS = t_{m,k} \circ t_{n,m},
= \left(\frac{1}{(m-k)!} \cdot t_{k+1} \circ \cdots \circ t_m\right) \cdot \left(\frac{1}{(n-m)!} \cdot t_{m+1} \circ \cdots \circ t_n\right),
= \frac{1}{(m-k)!(n-m)!} \cdot t_{k+1} \circ \cdots \circ t_n,
\]
\[
RHS = \left(\frac{n-k}{n-m}\right) \cdot t_{n,k},
= \frac{(n-k)!}{(n-m)!(m-k)!} \cdot \frac{1}{(n-k)!} \cdot t_{k+1} \circ \cdots \circ t_n,
= \frac{1}{(n-m)!(m-k)!} \cdot t_{k+1} \circ \cdots \circ t_n,
= LHS.
\]

Writing \(A_n = H_i(\text{Conf}_n(\mathcal{X}); \mathbb{Z})\) allows one to apply the lemma from Dold (Lemma 3.4.2) with,
\[
B_n := \text{Coker}(i_{n-1} : H_i(\text{Conf}_{n-1}(\mathcal{X}); \mathbb{Z}) \to H_i(\text{Conf}_n(\mathcal{X}); \mathbb{Z})).
\]
The conclusion of the lemma is that
\[
id \oplus t_{n,n-1} \oplus t_{n,n-2} \oplus \cdots \oplus t_{n,0} : A_n \to B_n \oplus B_{n-1} \oplus \cdots \oplus B_0,
\]
is an isomorphism, and thus the \(i_n\) as defined earlier, are split injective.
3.5 Main Result on Homological Injectivity

The work in this section culminates with the following theorem about homological injectivity on orbifolds.

**Theorem 3.5.1.** Let $X$ be an orbifold which is the interior of an orbifold with boundary of dimension at least 2. Assume that there exists a collar in $X$ around a closed full sub-orbifold of $\partial X$ as described in Section 3.1. Then there exists a map which adds a point near the boundary of $X$ to a configuration,

$$i : \text{Conf}_n X \to \text{Conf}_{n+1} X,$$

such that the induced map on integral homology is injective,

$$i_* : H_*(\text{Conf}_n X) \hookrightarrow H_*(\text{Conf}_{n+1} X).$$
Chapter 4

A Quasi-Fibration Criterion for Topological Groupoids

This chapter focusses on finding a quasi-fibration criterion on the category of topological groupoids. The motivation behind this is to use the criterion in a calculation of the limiting homology of configurations on certain orbifolds as the number of points tends to infinity.

Section 4.1 is a warm up with relevant definitions and contains a description of the pre-image of a point under a map between classifying spaces.

Once the background work is established, the task becomes working with the homotopy type of the fibre over a point in the classifying space. In Section 4.2, two homotopies are described on the two-sided bar construction which, when used together, give a nice reduction of the problem. After a bit of mundane condition checking, it becomes clear that the pre-image of a point is simply homotopic to the classifying space of some $G$. Here, $G$ is the pre-image of a single point in the object space of the target topological groupoid. Specifically, \[(Bf)^{-1}(Y_0 \leftarrow \cdots \leftarrow Y_n, t) \simeq BG,\]

where $G$ is the topological groupoid with objects \(\{X \mid f(X) = Y_0\}\) and morphisms \(\{g \mid f(g) = id_{Y_0}\}\) (see Corollary 4.2.6).

Moving into Section 4.3, more restrictions are imposed on the map between topological groupoids in order to give a criterion for when maps between topological groupoids are quasi-fibrations (that is, a quasi-fibration on the induced classifying space map). The condition here is that the classifying space map restricted to the pre-image of the object space in the target is itself a quasi-fibration.

Once the quasi-fibration criterion has been deduced, Section 4.4 makes use
of this previous work and applies it to certain configuration spaces on orbifolds. The result obtained is that, in certain circumstances, an orbifold map

\[ \text{Conf}(\mathcal{M}, \mathcal{N}) \to \text{Conf}(\mathcal{M}, \mathcal{N} \cup \mathcal{M}') \]

satisfies the conditions found previously and is therefore a quasi-fibration. In addition, the fibre of the map induced on classifying spaces is shown to be \( B\text{Conf}(\mathcal{M}', \mathcal{M}' \cap \mathcal{N}) \).

### 4.1 Background

The criterion found is in regards to topological groupoids, before applying it to a specific example on orbifolds (which are thought of by their groupoid definition). Rather than worry about all the technical details, one can intuitively think of a topological groupoid \( \mathcal{X} \) as an object space \( X_0 \) with a space of (invertible) arrows \( X_1 \) between objects. Full definitions can be found in Chapter 2. The following is made up of the extra definitions and observations which do not appear earlier in this document or are important enough to restate, and are required knowledge in the latter parts of this chapter.

Recall from an earlier chapter, the definitions of left and right \( G \) spaces.

**Definition 4.1.1** ([ALR07] Definition 2.14). Let \( G \) be a topological groupoid. A left \( G \)-space is a topological space \( E \) with an action of \( G \). The action comes in two parts,

1. an anchor map \( \pi : E \to G_0 \), and
2. an action map \( \mu : G_1 \times_{G_0} E \to E \), which is defined on \((g, e)\) when \( \pi(e) = s(g) \) (also written \( \mu(g, e) = g \cdot e \)).

The action satisfies the usual identities for an action, \( \pi(h \cdot e) = t(h) \), \( 1_x \cdot e = e \) and \( g \cdot (h \cdot e) = (gh) \cdot e \) when \( x \xrightarrow{h} y \xrightarrow{g} z \) in \( G_1 \) and \( e \in E \) with \( \pi(e) = x \).

**Definition 4.1.2.** Let \( G \) be a topological groupoid. A right \( G \)-space is a topological space \( E \) with an action of \( G \). The action comes in two parts,

1. an anchor map \( \pi : E \to G_0 \), and
2. an action map \( \mu : E \times_{G_0} G_1 \to E \), which is defined on \((e, g)\) when \( \pi(e) = t(g) \) (also written \( \mu(e, g) = e \cdot g \)).

The action satisfies the usual identities for an action, \( \pi(e \cdot g) = s(g), e \cdot 1_z = e \) and \( (e \cdot g) \cdot h = e \cdot (gh) \) when \( x \xrightarrow{h} y \xrightarrow{g} z \) in \( G_1 \) and \( e \in E \) with \( \pi(e) = z \).
4.1. BACKGROUND

**Definition 4.1.3.** Let $\mathcal{G}$ be a topological groupoid, $X$ a right $\mathcal{G}$-space and $Y$ a left $\mathcal{G}$-space with anchor and action maps,

\[
\pi_X : X \to G_0, \\
x \mapsto s(x),
\]

\[
\pi_Y : Y \to G_0, \\
y \mapsto t(y),
\]

\[
\mu_X : X \times_{G_0} G_1 \to X, \\
(x, g) \mapsto x \circ g,
\]

\[
\mu_Y : G_1 \times_{G_0} Y \to Y, \\
(y, g) \mapsto g \circ y.
\]

The two-sided bar construction $B(X, \mathcal{G}, Y)$ is

\[
B(X, \mathcal{G}, Y) = \bigsqcup_{n \geq 0} B_n(X, \mathcal{G}, Y) \times \Delta^n / \sim,
\]

\[
B_n(X, \mathcal{G}, Y) = (X \times_{G_0} G_1 \times_{G_0} G_1 \times_{G_0} \cdots \times_{G_0} G_1 \times_{G_0} Y)
\]

where elements are of the form $((x, g_1, \ldots, g_n, y), (t_0, \ldots, t_n))$ with $s(g_1) = t(g_{i+1})$, $\pi_X(x) = t(g_1)$, $\pi_Y(y) = s(g_n)$, $t_i \geq 0$, $\sum t_i = 1$ and $\sim$ is the equivalence relation generated by,

1. $((x, g_1, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n, y), (t_0, \ldots, t_n))$

   \[\sim ((x, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n, y), (t_0, \ldots, t_{i-2}, t_{i-1} + t_i, t_{i+1}, \ldots, t_n))\]

2. $((x, g_1, \ldots, g_n, y), (0, t_1, \ldots, t_n))$

   \[\sim ((\mu_X(x, g_1), g_2, \ldots, g_n, y), (t_1, \ldots, t_n))\]

3. $((x, g_1, \ldots, g_n, y), (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n))$

   \[\sim ((x, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n, y), (t_0, \ldots, t_{i-1}, t_{i+1}, t_{i+2}, \ldots, t_n))\]

4. $((x, g_1, \ldots, g_n, y), (t_0, \ldots, t_{n-1}, 0))$

   \[\sim ((x, g_1, \ldots, g_{n-1}, \mu_Y(g_n, y)), (t_0, \ldots, t_{n-1})).\]

Write $BG := B(\ast, \mathcal{G}, \ast)$ for the regular bar construction of the groupoid $\mathcal{G}$. Something to note at this point is that the above definition differs from what was being called $BG$ in Chapter 4 (the fat classifying space). The difference between the two is the first condition above, which allows the removal of identity arrows, does not appear in the previous definition. However, the use of the same notation for both is vindicated by each of these constructions defining spaces which are homotopic (see [Seg74, Proposition A.1] for reference).
Definition 4.1.4. Let $f : C \to C'$ be a morphism between categories. The induced map on classifying spaces is defined to be:

$$Bf : BC \to BC', \quad ((*, g_1, \ldots, g_n, *), t) \mapsto ((*, f(g_1), \ldots, f(g_n), *), t).$$

As this chapter is about quasi-fibrations, a description of the fibre over a point in a simplex $Y = Y_0 \xleftarrow{h_1} Y_1 \xleftarrow{h_2} \cdots \xleftarrow{h_n} Y_n$ in $BC'$ is important. It would be unwise to only consider $n$-simplices in $BC$ which map down to $Y$. Indeed, while there do exist $n$-simplices in $BC$ which map down to $Y$, there may be more. One also needs to consider simplices which have dimension greater than $n$ and end up with identity arrows introduced upon being mapped by $Bf$. Using the relevant equivalence relation from the two-sided bar construction, these identity arrows may be dropped to obtain a simplex of dimension $n$.

Proposition 4.1.5. Suppose that $(Y, t) = (Y_0 \xleftarrow{h_1} \cdots \xleftarrow{h_n} Y_n, t) \in B_n C'$ with $t \in \Delta^n$. The fibre over $(Y, t)$ can be described as,

$$(Bf)^{-1}(Y, t) = \bigsqcup_l \{(X, s) \in B_l C \times \Delta^l \mid Bf(X, s) \sim (Y, t)\} / \sim,$$

where $\sim$ is the equivalence relation defined above for classifying spaces. Elements of $(Bf)^{-1}(Y, t)$ are of the form,

$$\left(X^0 \xleftarrow{g_0^1} \cdots \xleftarrow{g_0^n} X^0, \cdots, X^n \xleftarrow{g_n^1} \cdots \xleftarrow{g_n^n} X^n, s\right),$$

such that $X$ and $s$ satisfy the following,

1. $f(X^i_j) = Y_i$;
2. $f(g^i_j) = id_{Y_i}$;
3. $f(\eta_i) = h_i$; and
4. $s \in t_0 \Delta^{l_0} \times t_1 \Delta^{l_1} \times \cdots \times t_n \Delta^{l_n}$.

This description could be considered opaque and a better description of this $X$ might be preferred. An iterated bar construction can be used to obtain this ‘preferred’ description.

Proposition 4.1.6. Let $(Y, t) \in B_n C'$ with $t \in \Delta^n$. Then the fibre over $(Y, t) = (Y_0 \xleftarrow{h_1} \cdots \xleftarrow{h_n} Y_n, t)$ is,

$$(Bf)^{-1}(Y, t) = B(\ast, G^0, B(E_1, G^1, B(E_2, G^2, B(\cdots, B(E_n, G^n, \ast))))),$$

where the $G^i$ and $E_i$ are defined as follows,
4.1. BACKGROUND

1. $G^i \subseteq C$, $G^i_0 = f^{-1}(Y_i)$, $G^i_1 = f^{-1}(id_{Y_i})$.

2. $E_i = \{ \alpha \in C_1 \mid f(\alpha) = h_i \}$.

3. anchor maps

\[ \pi^i_i : E_i \rightarrow G^i_0, \quad \pi^i_{i+1} : E_{i+1} \rightarrow G^i_0, \]

\[ \alpha \mapsto s(\alpha), \quad \alpha \mapsto t(\alpha), \]

and

4. action maps

\[ \mu^i_i : E_i \times_{G^i_0} G^i_1 \rightarrow E_i, \quad \mu^i_{i+1} : G^i_1 \times_{G^i_0} E_{i+1} \rightarrow E_{i+1}, \]

\[ (\alpha, \beta) \mapsto \alpha \circ \beta, \quad (\beta, \alpha) \mapsto \beta \circ \alpha. \]

**Sketch of Proof.** The proof will show the idea in the case that $n = 1$, higher values of $n$ can be obtained iteratively. The point the fibre is taken over is $(Y_0 \xrightarrow{h} Y_1, (t_0, t_1))$.

In the first description of the fibre over $(Y, t)$, objects are of the form

\[ (X^0_0 \underset{\cdots \underset{g^0_{l_0}}{\cdots \underset{g^1_0}{\cdots \underset{g^1_{l_1}}{\cdots}}}}{\cdots \underset{g^0_{l_0}}{\cdots \underset{g^1_0}{\cdots \underset{g^1_{l_1}}{\cdots}}}} X^i_0, s), \]

where

1. $f(X^i_0) = Y_i$;
2. $f(g^i_j) = id_{Y_i}$;
3. $f(\eta) = h$; and
4. $s \in t_0 \Delta^{l_0} \times t_1 \Delta^{l_1},$

and two points are identified if they differ by face or degeneracy maps.

One can observe that the first $l_0$ arrows, above,

\[ X^0_0 \underset{\cdots \underset{g^0_{l_0}}{\cdots \underset{g^1_0}{\cdots \underset{g^1_{l_1}}{\cdots}}}}{\cdots \underset{g^0_{l_0}}{\cdots \underset{g^1_0}{\cdots \underset{g^1_{l_1}}{\cdots}}}} X^i_0 \]

is $l_0$ composable arrows in $G^0$ and taking $s^0 := (s_0, \ldots, s_n)/t_0 \in \Delta^{l_0}$. Giving,

\[ (X^0_0 \underset{\cdots \underset{g^0_{l_0}}{\cdots \underset{g^1_0}{\cdots \underset{g^1_{l_1}}{\cdots}}}}{\cdots \underset{g^0_{l_0}}{\cdots \underset{g^1_0}{\cdots \underset{g^1_{l_1}}{\cdots}}}} X^i_0, s^0), \]

a point in $BG^0$. A similar argument gives the same result for the last $l_1$ arrows.

The $\eta$ map, which is chosen from all the arrows over $h$, joins the $BG^0$ and $BG^1$ together. This is the motivation for the choice of $E$ being the space of
arrows in $C_1$ which lie over $h$ in $C'_1$. The anchor maps are chosen to ensure that
the maps in the full sequence compose, the action maps are chosen so that the
$E$ part can take part in the face and degeneracy maps.

Though this proof isn’t rigorous, this should hopefully reassure the reader
that the fibre over $(Y, t)$ is indeed the space

$$B(*, G^0, B(E, G^1, *))$$

In order to simplify this description of the fibre over a point further, a ho-
motopy equivalent description will be found which is far simpler. In order to do
this, the idea of a homotopy in the category of simplicial objects is needed. Us-
ing Peter May’s *Simplicial Objects in Algebraic Topology* [May93] as a reference
the concept of such a homotopy follows.

**Definition 4.1.7** ([May93]). Let $f, g : X \rightarrow Y$ be maps in $ST$, simplicial
objects in a category $T$. A homotopy $h : f \simeq g$ consists of maps $h_i : X_q \rightarrow Y_{q+1}$,
$0 \leq i \leq q$ such that

$$\partial_0 h_0 = f_q,$$
$$\partial_{q+1} h_q = g_q,$$
$$\partial_i h_j = \begin{cases} h_{j-1} \partial_i, & \text{if } i < j, \\ \partial_j h_{j-1}, & \text{if } i = j > 0, \\ h_j \partial_{i-1}, & \text{if } i > j + 1, \end{cases}$$
$$s_i h_j = \begin{cases} h_{j+1} s_i, & \text{if } i \leq j, \\ h_j s_{i-1}, & \text{if } i > j. \end{cases}$$

Finally, before moving on, recall the definition of homotopy fibre (Definition
4.3.1) and quasi-fibration (Definition 4.3.11).

If $f : X \rightarrow Y$ is a continuous map between topological spaces, the *homotopy
fibre* over a fixed base-point $y \in Y$ is the space

$$hofib_y(f) := \{(x, \gamma) | x \in X, \gamma : I \rightarrow Y, \gamma(0) = f(x), \gamma(1) = y\}.$$  

The map $f$ is called a *quasi-fibration* if whenever $y \in Y$ then the canonical
map between the fibre over $y$ and the homotopy fibre over $y$,

$$f^{-1}(y) \rightarrow hofib_y(f),$$
$$z \mapsto (z, c_y),$$
is a homotopy equivalence, where \( c_y \) is the constant path at \( y \).

## 4.2 Homotopy Type of the Fibres

To simplify the iterated bar construction description of the pre-image of a point, the following two homotopy equivalences will be shown with some restrictions:

1. \( B(?, G^i, G^i_0) \simeq B(?, G^i, *) \) for every \( ? \); and

2. \( B(E_i, G^i, *) \simeq G^i_0 \).

Putting these together and iterating gives the homotopy equivalence,

\[
B(\ast, G^0, B(E_1, G^1, B(\cdots, B(E_{n-1}, G^{n-1}, B(E_n, G^n, *)) \cdots))) \simeq B G^0.
\]

The first of these two is simple enough to show.

**Proposition 4.2.1.** Let \( G \) be a topological groupoid, and let \( X \) be any right \( G \)-space. Then

\[
B(X, G, G_0) \simeq B(X, G, *),
\]

where \( G_0 \) is the left \( G \)-space with anchor map

\[
\pi : G_0 \to G_0,
\]

\[
x \mapsto x,
\]

and action map

\[
\mu : G_1 \times_{G_0} G_0 \to G_0,
\]

\[
(\beta, x) \mapsto t(\beta).
\]

**Proof.** This is fairly straight forward as:

\[
B(X, G, G_0) = X \times_{G_0} G_1 \times_{G_0} G_1 \times_{G_0} \cdots \times_{G_0} G_1 \times_{G_0} G_0,
\]

\[
= X \times_{G_0} G_1 \times_{G_0} G_1 \times_{G_0} \cdots \times_{G_0} G_1,
\]

\[
= B(X, G, *).
\]

To get from the first to the second line, notice the following. The first line, is tuples of composable arrows from \( G_1 \) where the right-most arrow has source in \( G_0 \). However, seeing as all arrows in \( G_1 \) have source in \( G_0 \), the \( \times_{G_0} G_0 \) part is superfluous and can be dropped, as in the second line.

\[\square\]
The second of the two homotopy equivalences required takes substantially more work and uses the description of a homotopy in the category on simplicial objects in a category as described in the previous section.

**Proposition 4.2.2.** Let \( f : C \to C' \) be a map between topological groupoids. Let \( h : Y_0 \leftarrow Y_1 \) be an arrow in \( C' \) and let

- \( G^1 \) be the sub-category of \( C \) with objects \( \{ X \mid f(X) = Y_i \} \) and morphisms \( \{ g \mid f(g) = id_{Y_i} \} \) for \( i = 0, 1 \); and

- \( E \subseteq C_1 \) be the space of arrows \( \{ g \in C_1 \mid f(g) = h \} \), \( E \) is a right \( G^1 \) space with
  - anchor map
  \[
  \pi : E \to G^1_0,
  \]
  \[
  \alpha \mapsto s(\alpha),
  \]
  - action map
  \[
  \mu : E \times_{G^1_1} G^1_1 \to E,
  \]
  \[
  (\alpha, \beta) \mapsto \alpha \circ \beta.
  \]

If there is a continuous section of the target map \( t : E \to G^0_0 \), then there is a homotopy equivalence,

\[
B(E, G^1, \ast) \simeq G^0_0.
\]

**Proof.** The end goal is to show that \( B(E, G^1, \ast) \simeq G^0_0 \). Note that the continuous section assumption says that the map \( t : E \to G^0_0 \) splits. To get the required result, the proof will define two simplicial maps and show they are homotopy inverses. One is therefore required to think of \( G^0_0 \) as a simplicial complex. Define \( X_\bullet \) to be the constant simplicial space with \( X_n = G^0_0 \) and all face and degeneracy maps the identity. Then the geometric realisation of this simplicial space is \( |X_\bullet| = G^0_0 \). The two simplicial maps which need to be defined are,

\[
\alpha_n : B_n(E, G^1, \ast) \to X_n,
\]
\[
\beta_n : X_n \to B_n(E, G^1, \ast),
\]
such that \( \alpha_n \circ \beta_n \simeq id \) and \( \beta_n \circ \alpha_n \simeq id \).

Using the assumption of a continuous splitting of the map,

\[
t : E \to G^0_0,
\]
4.2. HOMOTOPY TYPE OF THE FIBRES

pick a continuous section,

\[ \iota : G_0^0 \to E. \]

Since \( \iota \) is a section, one can note that \( t \circ \iota = id_{G_0^0} \). Define the two sets of maps \( \alpha_n \) and \( \beta_n \) to be

\[
\alpha_n : B_n(E, G^1, *) \to X_n = G_0^0,
(a, b_1, \ldots, b_n, *) \mapsto t(a),
\]

and

\[
\beta_n : X_n = G_0^0 \to B_n(E, G^1, *),
c \mapsto (\iota(c), 1_{s(i(c))}, \ldots, 1_{s(i(c))}, *).\]

One can check that these are indeed simplicial maps by showing that they commute with face and degeneracy maps. For example, observe that

\[
\partial_i \alpha_n(a, b_1, \ldots, b_n, *) = \partial_i t(a),
= t(a),
= \alpha_{n-1}(a, b_1, \ldots, b_{i-1}, b_i b_{i+1}, b_{i+2}, \ldots, b_n, *),
= \alpha_{n-1} \partial_i(a, b_1, \ldots, b_n, *).
\]

The other combinations are left as an exercise for the reader.

The next step is to show both that \( \alpha_n \circ \beta_n \simeq id \) and \( \beta_n \circ \alpha_n \simeq id \). The first of these is straight forward, as

\[
\alpha_n \circ \beta_n(c) = \alpha_n(\iota(c), 1_{s(i(c))}, \ldots, 1_{s(i(c))}, *),
= t(\iota(c)),
= c,
\]

by the assumption that \( \iota \) is a continuous section of \( t \).

This leaves the other half of showing that \( \beta_n \circ \alpha_n \simeq id \). A homotopy in the category of simplicial objects is needed to do this (see Definition 4.1.7). Define the homotopy map to be

\[
\psi : B_n(E, G^1, *) \to B_{n+1}(E, G^1, *),
(a, b_1, \ldots, b_n, *) \mapsto (a, b_1, \ldots, b_i, b_{i+1} \circ \cdots \circ b_n \circ (\pi \circ b_1 \circ \cdots \circ b_i)^{-1}, 1, 1, \ldots, 1, *)
= (a, b_1, \ldots, b_i, (\pi \circ b_1 \circ \cdots \circ b_i)^{-1}, 1_{t(\pi)}, 1_{t(\pi)}, \ldots, 1_{t(\pi)}, *).
\]
where,
\[ \overline{a} := (\iota(t(a)))^{-1} \circ a. \]

Note that \( \overline{a} \) is well defined as,
1. \( s((\iota(t(a)))^{-1}) = t(\iota(t(a))) = t(a) \) by \( \iota \) being a section of \( t \); and
2. \( s(a) \in f^{-1}(s(h)) = f^{-1}(Y_1) = G_0^1 \) and \( t((\iota(t(a)))^{-1}) = s(\iota(t(a))) \in s(E) \subseteq G_1^0 \). So \( \overline{a} \in G_1^1 \).

The choice of the above \( \psi_n \) as well as \( \alpha \) and \( \beta \) were heavily influenced by reading Proposition 4.1 of [WW15]. The map \( \psi_n \) introduces an ‘extra degeneracy’ which is what allows the homotopy to be defined.

In order to show that \( \psi \) defines a homotopy between the maps \( \beta \circ \alpha \) and \( id \), the following conditions need to be checked,

\[
\begin{align*}
\partial_0 \psi_0 &= \beta_n \circ \alpha_n, \\
\partial_{n+1} \psi_n &= id_n, \\
\partial_j \psi_k &= \begin{cases} \\
\psi_{k-1} \partial_j, & \text{if } j < k, \\
\psi_k \partial_{k-1}, & \text{if } j = k > 0, \\
\psi_k \partial_j - 1, & \text{if } j > k + 1, \\
\end{cases} \\
s_j \psi_k &= \begin{cases} \\
\psi_{k+1} s_j, & \text{if } j \leq k, \\
\psi_k s_j - 1, & \text{if } j > k. \\
\end{cases}
\end{align*}
\]

Firstly, showing that \( \partial_0 \psi_0 = \beta_n \circ \alpha_n \):

\[
\begin{align*}
\partial_0 \psi_0(a, b_1, \ldots, b_n, *) &= \partial_0(a, \overline{\pi}^{-1}, 1(t), 1(t), \ldots, 1(t), *), \\
&= (a \circ \overline{\pi}^{-1}, 1(t), 1(t), \ldots, 1(t), *), \\
&= (a \circ [\iota(t(a))]^{-1} \circ a]^{-1}, 1(t), 1(t), \ldots, 1(t), *), \\
&= (a \circ a^{-1} \circ \iota(t(a)), 1s(\iota(t(a))), 1s(\iota(t(a))), \ldots, 1s(\iota(t(a))), *), \\
&= (\iota(t(a)), 1s(\iota(t(a))), 1s(\iota(t(a))), \ldots, 1s(\iota(t(a))), *), \\
&= \beta_n(t(a)), \\
&= \beta_n \circ \alpha_n(a, b_1, \ldots, b_n, *).
\end{align*}
\]

Since

\[
\begin{align*}
t(\overline{a}) &= t((\iota(t(a)))^{-1} \circ a), \\
&= t((\iota(t(a)))^{-1}), \\
&= s(\iota(t(a))).
\end{align*}
\]
4.2. HOMOTOPY TYPE OF THE FIBRES

Checking the next condition, \( \partial_{n+1} \psi_n = id_n \), see that

\[
\partial_{n+1} \psi_n(a, b_1, \ldots, b_n, *) = \partial_{n+1}(a, b_1, \ldots, b_n, (\overline{\pi} \circ b_1 \circ \cdots \circ b_n)^{-1}, *),
\]
\[
= (a, b_1, \ldots, b_n, *),
\]
\[
= id_n(a, b_1, \ldots, b_n, *),
\]
as required.

The remainder of the proof will be showing the other conditions, the reader should feel free to skip to the end unless they are in desperate need of some character building.

Check that \( \partial_j \psi_k = \psi_{k-1} \partial_j \) when \( j < k \):

\[
\partial_j \psi_k(a, b_1, \ldots, b_n, *)
\]
\[
= \partial_j(a, b_1, \ldots, b_k, (\overline{\pi} \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
\]
\[
= (a, b_1, \ldots, b_{j-1}, b_j \circ b_{j+1}, b_{j+2}, \ldots, b_k, (\overline{\pi} \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
\]
\[
= \psi_{k-1}(a, b_1, \ldots, b_{j-1}, b_j \circ b_{j+1}, b_{j+2}, \ldots, b_n, *),
\]
\[
= \psi_{k-1} \partial_j(a, b_1, \ldots, b_n, *).
\]

Check that \( \partial_j \psi_k = \partial_k \psi_{k-1} \) when \( j = k > 0 \):

\[
\partial_j \psi_k(a, b_1, \ldots, b_n, *)
\]
\[
= \partial_j(a, b_1, \ldots, b_k, (\overline{\pi} \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
\]
\[
= (a, b_1, \ldots, b_{k-1}, b_k \circ (\overline{\pi} \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
\]
\[
= (a, b_1, \ldots, b_{k-1}, (\overline{\pi} \circ b_1 \circ \cdots \circ b_{k-1})^{-1}, 1, 1, \ldots, 1, *),
\]
\[
= \partial_k(a, b_1, \ldots, b_{k-1}, (\overline{\pi} \circ b_1 \circ \cdots \circ b_{k-1})^{-1}, 1, 1, 1, \ldots, 1, *),
\]
\[
= \partial_k \psi_{k-1}(a, b_1, \ldots, b_n, *).
\]

Note that in the second to last line there is an extra identity arrow inserted the \( k + 1 \) position.

Check that \( \partial_j \psi_k = \psi_{k-1} \partial_j \) when \( j > k + 1 \):

\[
\partial_j \psi_k(a, b_1, \ldots, b_n, *)
\]
\[
= \partial_j(a, b_1, \ldots, b_k, (\overline{\pi} \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
\]
\[
= (a, b_1, \ldots, b_{k-1}, b_k, (\overline{\pi} \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
\]
\[
= \psi_k(a, b_1, \ldots, b_{j-2}, b_{j-1} \circ b_j, b_{j+1}, \ldots, b_n, *),
\]
\[
= \psi_k \partial_{j-1}(a, b_1, \ldots, b_n, *).
\]

Note that in the third line, the two identity arrows in the \( j \) and \( j + 1 \) positions
have been composed, leaving one less \( 1 t(\pi) \).

Check that \( s_j \psi_k = \psi_{k+1} s_j \) when \( j \leq k \):

\[
s_j \psi_k (a, b_1, \ldots, b_n, *) = s_j (a, b_1, \ldots, b_k, (\pi \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
= (a, b_1, \ldots, b_j, 1_{s(b_j)}, b_{j+1}, \ldots, b_k, (\pi \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
= (a, b_1, \ldots, b_j, 1, b_{j+1}, \ldots, b_k, (\pi \circ b_1 \circ \cdots \circ b_j \circ 1 \circ b_{j+1} \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
= \psi_{k+1} (a, b_1, \ldots, b_j, 1, b_{j+1}, \ldots, b_n, *)
= \psi_{k+1} s_j (a, b_1, \ldots, b_n, *).
\]

Check that \( s_j \psi_k = \psi_k s_{j-1} \) when \( j > k \):

\[
s_j \psi_k (a, b_1, \ldots, b_n, *)
= s_j (a, b_1, \ldots, b_k, (\pi \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
= (a, b_1, \ldots, b_k, (\pi \circ b_1 \circ \cdots \circ b_k)^{-1}, 1, 1, \ldots, 1, *),
= \psi_k (a, b_1, \ldots, b_{j-1}, 1_{t(b_j)}, b_j, \ldots, b_n, *)
= \psi_k s_{j-1} (a, b_1, \ldots, b_n, *).
\]

Note that on the third line there is an extra identity arrow in the \( j + 1 \) position.

All the conditions required for \( \psi \) to be a homotopy have been checked, so it is indeed a homotopy between \( \beta \circ \alpha \rightarrow id \). Putting this together with the previous observation that \( \alpha \circ \beta \simeq id \), gives the result that

\[
B(E, G^1, *) \simeq G_0^0.
\]

\[\square\]

**Remark 4.2.3.** The proof (above) of Proposition 4.2.2 also shows that \( G_0^0 \) is a deformation retraction of \( B(E, G^1, *) \).

The next step is to apply these two propositions to the description of the pre-image of a point from \( BC' \) under \( Bf \). Putting Propositions 4.2.1 and 4.2.2 together gives the following Corollary.

**Corollary 4.2.4.** If \( t : E_i \rightarrow G_0^{i-1} \) splits for every \( i \in \{1, \ldots, n\} \) then

\[
B(*, G^0, B(E_1, G^1, B(\cdots, B(E_{n-1}, G^{n-1}, B(E_n, G^n, *)) \cdots))) \simeq B G_0^0.
\]

**Proof.** This is clear using the previous observations. First apply Proposition
$B(E^n, G^n, *) \simeq G^n_{0^{-1}},$

then follow with Proposition 4.2.2

$B(E^{n-1}, G^{n-1}, G^n_0) \simeq B(E^{n-1}, G^{n-1}, *),$

$\simeq G^n_{0^{-2}},$

and continue. Eventually the problem gets reduced to,

$B(*, G^0, B(E^1, G^1, G^1_0)) \simeq B(*, G^0, G^0_0),$

$\simeq B(*, G^0, *),$

$\simeq B G^0,$

as required.

Corollary 4.2.5. Let $f : C \to C'$ be a map of topological groupoids. Recall that a non-degenerate point in $B C'$ is of the form,

$(Y_0 \leftarrow h_{1} \leftarrow Y_1 \leftarrow \cdots \leftarrow h_{n} \leftarrow Y_n, t),$

with $t \in \text{int}(\Delta^n)$. Write $E_i \subseteq C_1$ for the space of arrows $\{g \mid f(g) = h_i\}$ and $G^i$ for the topological groupoid with objects $\{X \in C_0 \mid f(X) = Y_i\}$ and morphisms $\{g \in C_1 \mid f(g) = \text{id}_{Y_i}\}$. If the target maps $t : E_i \to G^i_{0^{-1}}$ split for every $i \in \{1, \ldots, n\}$, then

$(Bf)^{-1}(Y_0 \leftarrow h_{1} \leftarrow \cdots \leftarrow h_{n} \leftarrow Y_n, t) \simeq B G^0.$

Corollary 4.2.6. Let $f : C \to C'$ be a map of topological groupoids. Suppose that whenever there is an arrow $h : Z_0 \leftarrow Z_1$ in $C'$ then there exists a continuous splitting of the target map $t : E \to H^0_0$, where

- $E \subseteq C_1$ is the subspace $\{g \mid f(g) = h\}$; and
- $H^0_0 \subseteq C_0$ is the subspace $\{X \mid f(X) = Z_0\}.$

Then the preimage of any non-degenerate point from $B C'$, $(Y_0 \leftarrow \cdots \leftarrow Y_n, t)$ satisfies

$(Bf)^{-1}(Y_0 \leftarrow \cdots \leftarrow Y_n, t) \simeq B G^0,$

where $G^0$ is the topological groupoid with objects $\{X \in C_0 \mid f(X) = Y_0\}$ and morphisms $\{g \in C_1 \mid f(g) = \text{id}_{Y_0}\}.$
4.3 The Quasi-Fibration

The search is still on for what conditions are needed on a map \( f : C \to C' \) in order to be able to say that the induced map on classifying spaces, \( Bf : BC \to BC' \), is a quasi-fibration. This section is concerned with showing that the problem reduces to simply requiring that there is a quasi-fibration over the object space of \( C' \). The first port of call is thus to show that if it is assumed that there is a quasi-fibration over the object space of \( C' \), then the entire map \( Bf : BC \to BC' \) is indeed a quasi-fibration.

**Theorem 4.3.1.** Let \( F_p \subseteq BC' \) be the \( p \)-skeleton of the classifying space. That is,

\[
F_p := \bigcup_{0 \leq n \leq p} C'_n \times \Delta^n / \sim,
\]

where \( \sim \) is the usual equivalences for a classifying space. The \( F_p \) give a filtration of \( BC' \):

\[
F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{p-1} \subseteq F_p \subseteq \cdots.
\]

Assume that the map \( f : C \to C' \) satisfies the conditions of Corollary 4.2.6 and assume that the map

\[
(Bf)^{-1}(F_0) \xrightarrow{Bf} F_0
\]

is a quasi-fibration. Then the entire map

\[
Bf : BC \to BC'
\]

is a quasi-fibration.

**Proof.** Recalling the definition of a quasi-fibration, the task is to show that if \( (Y, t) \in BC' \) then \( Bf_{(Y, t)} \simeq hofib_{(Y, t)}(Bf) \), where \( Bf_{(Y, t)} \) is the fibre over the point \( (Y, t) \) and \( hofib_{(Y, t)}(Bf) \) is the homotopy fibre over the point \( (Y, t) \).

In order to show this homotopy equivalence the following diagram will be of use:

\[
\begin{array}{ccc}
(Bf)_{(Y, t)} & \xrightarrow{} & hofib_{(Y, t)}(Bf) \\
\downarrow & & \downarrow \\
(Bf)_{(Y_0, 1)} & \xrightarrow{} & hofib_{(Y_0, 1)}(Bf).
\end{array}
\]

Showing the full map \( Bf \) is a quasi-fibration, is precisely showing that the top
arrow is a homotopy equivalence for every \((Y, t)\). In order to show this, the proof first asserts that the other three maps in the diagram are homotopy equivalences, then it ends by showing that the entire diagram is homotopy commutative. Once this has been done, the result will have been achieved.

Investigate the diagram above.

The arrow along the bottom of the diagram is the map,

\[
(Bf)_{(Y_0, 1)} \to hofib_{(Y_0, 1)}(Bf),
\]

\[
z \mapsto (c_{Bf(z)}, z),
\]

where \(c_{Bf(z)}\) is the constant path at \(Bf(z)\). By the assumption of the theorem, that the restriction of \(Bf\) over the 0-skeleton is a quasi-fibration, immediately gives that this map is a homotopy equivalence.

The arrow down the left of the diagram, which is made up from the homotopy on the two-sided bar construction (see Corollary 4.2.6), comes from iterating the maps

\[
((*, g^0_1, \ldots, g^0_p, \eta_1, \ldots, \eta_p, g^1_p, \ldots, g^n_p, *), (t^0, \ldots, t^n))
\]

\[
\mapsto ((*, g^0_1, \ldots, g^0_p, \eta_1, \ldots, \eta_{n-1}, g^{n-1}_p, \ldots, g^{n-1}_0, 1_{t(Y_0)}, 1, \ldots, 1, *), (t^0, \ldots, t^n)),
\]

\[
\sim ((*, g^0_1, \ldots, g^0_p, \eta_1, \ldots, \eta_{n-1}, g^{n-1}_p, \ldots, g^{n-1}_0, 1_{t(Y_0)}, *), (t^0, \ldots, t^{n-1}, \Sigma t^n)),
\]

\[
\sim ((*, g^1_1, \ldots, g^1_p, \eta_1, \ldots, \eta_{n-1}, g^{n-1}_p, \ldots, g^{n-1}_0, 1_{t(Y_0)}), (t^0, \ldots, t^{n-1}, t^{n-1}_{i_{n-1}}, t^n + \Sigma t^n)).
\]

Here, the first term is a point in the fibre over

\[
(Y, t) = (Y_0 \xleftarrow{h_1} \cdots \xleftarrow{h_n} Y_n, (t^0, \ldots, t^n)),
\]

this means that \(f(g^0_i) = \text{id}_{Y_i}, f(\eta_i) = h_i\) and \(t^i \in t^i \Delta^1\). The first-to-second line map is one of the steps in the homotopy. To get the third and fourth lines from this uses the equivalences in the definition of the bar construction, namely, the ‘if we have an identity arrow, then add up the corresponding co-ordinates in the simplex’ one. Iterating this to eliminate of all the \(\eta_j\) gives that the homotopy equivalence on the left of the diagram is the map:

\[
(Bf)_{(Y, t)} \to (Bf)_{(Y_0, 1)},
\]

\[
((g^1_1, \ldots, g^1_p, \eta_1, \ldots, \eta_p), (t^0, \ldots, t^p))
\]

\[
\mapsto ((g^1_1, \ldots, g^1_p), (t^0, \ldots, t^{n-1}_{i_{n-1}}, t^n_{i_{n-1}} + \Sigma t^n)).
\]

The arrow down the right of the diagram, is a homotopy equivalence if there
is a path from \((Y, t)\) to \((Y_0, 1)\). In which case, it is the map

\[ hofib_{(Y, t)}(Bf) \rightarrow hofib_{(Y_0, 1)}(Bf), \]
\[ (\gamma, w) \mapsto (q \circ \gamma, w), \]

where \(q\) is a fixed path from \((Y, t)\) to \((Y_0, 1)\). Since \(Y_0\) is a corner of the simplex \(Y\), such a path always exists, and thus the arrow down the right of the diagram is a homotopy equivalence.

The arrow along the top (which is the map this proof aims at showing is a homotopy equivalence) is the map,

\[ (Bf)_{(Y, t)} \rightarrow hofib_{(Y, t)}(Bf), \]
\[ z \mapsto (c_{Bf(z)}, z). \]

The diagram now looks as follows:

\[
\begin{array}{ccc}
(Bf)_{(Y, t)} & \rightarrow & hofib_{(Y, t)}(Bf) \\
\downarrow & \cong & \downarrow \\
(Bf)_{(Y_0, 1)} & \rightarrow & hofib_{(Y_0, 1)}(Bf).
\end{array}
\]

So, in order to show that \((Bf)_{(Y, t)} \rightarrow hofib_{(Y, t)}(Bf)\) is a homotopy equivalence, and therefore the quasi-fibration, all that remains to show is that the above diagram is homotopy commutative. To check for homotopy commutativity, one needs to check that there is a homotopy between the two possible ways of mapping from \((Bf)_{(Y, t)}\) to \(hofib_{(Y_0, 1)}(Bf)\). The two maps are

\[ (Bf)_{(Y, t)} \rightarrow hofib_{(Y_0, 1)}(Bf), \]
\[ z \mapsto (q, z), \quad \text{and} \]
\[ z \mapsto (c_{Bf(\tau(z))}, \tau(z)), \]
\[ = (c_{(Y_0, 1)}, \tau(z)), \]

where \(q\) is a chosen path from \((Y, t)\) to \((Y_0, 1)\) and \(\tau\) is the homotopy described above from \((Bf)_{(Y, t)}\) to \((Bf)_{(Y_0, 1)}\). (That is, the map down the left of the diagram). The path \(q\) needs to be picked correctly so that this works.

A homotopy between \((q, z)\) and \((c_{Bf(\tau(z))}, \tau(z))\) is needed, that is a contin-
4.3. THE QUASI-FIBRATION

uous map

\[ H : (Bf)(Y,t) \times I \to \text{hofib}(Y_0,1)(Bf), \]
\[ H_0(z) = (q, z), \]
\[ H_1(z) = (c(1,1), \tau(z)). \]

The idea behind the defining of this homotopy is as follows. Firstly, choosing the path \( q \) will be done so that it follows the path in the homotopy down the left of the diagram - that is, the homotopy:

\[ B(\ast, \mathcal{G}^0, B(E_1, \mathcal{G}^1, B(\cdots, B(E_{n-1}, \mathcal{G}^{n-1}, B(E_n, \mathcal{G}^n, \ast)) \cdots)) \simeq B\mathcal{G}^0. \]

Starting at a point in an \( n \)-simplex, the path \( q \) projects onto an \( (n-1) \)-simplex, then onto an \( (n-2) \)-simplex and repeats until it projects onto the 0-simplex \( Y_0 \). The homotopy on the path \( q \) is then simply shortening the path until it retracts onto \( (Y_0, 1) \) at \( H_1(z) \). However, each point in the homotopy needs to be a point in the homotopy fibre, and so far only a homotopy on the path \( q \) has been described. The second part of the proof constructs a homotopy on the point \( z \), so that it continuously moves in time with the endpoint of the homotopy on \( q \).

The approach will be to use the steps in the homotopy \( \tau \) to pick the path \( q \) from \( (Y, t) \) to \( (Y_0, 1) \) and achieve the result. Define the path \( q \) in \( n \) parts, \( q_i \) for \( i \in \{1, \ldots, n\} \). Each \( q_i \) will be a path from

\[ (Y_0 \leftarrow \cdots \leftarrow Y_i, (t^0, \ldots, t^{i-1}, t^i + \cdots + t^n)) \]

to

\[ (Y_0 \leftarrow \cdots \leftarrow Y_i, (t^0, \ldots, t^{i-2}, t^{i-1} + t^i + \cdots + t^n, 0)) \]
\[ \sim (Y_0 \leftarrow \cdots \leftarrow Y_{i-1}, (t^0, \ldots, t^{i-2}, t^{i-1} + t^i + \cdots + t^n)), \]

and is defined to be

\[ q_i(s) = (Y_0 \leftarrow \cdots \leftarrow Y_i, (t^0, \ldots, t^{i-2}, t^{i-1} + s(t^i + \cdots + t^n), (1 - s)(t^i + \cdots + t^n))). \]

Define the path \( q \) to be

\[ q = q_1 \circ \cdots \circ q_n, \]
which is a path from \((Y, t)\) to \((Y_0, 1)\), that is, the path

\[
q(s) = \begin{cases} 
q_n(s \times n), & s \in [0, \frac{1}{n}], \\
q_{n-1}((s \times n) - 1), & s \in [\frac{1}{n}, \frac{2}{n}], \\
\vdots & \\
q_{n-i}(s \times n) - i), & s \in [\frac{i}{n}, \frac{i+1}{n}], \\
\vdots & \\
q_1((s \times n) - n + 1), & s \in [\frac{n-1}{n}, 1]. 
\end{cases}
\]

Something that is also needed is a continuous path taking \(z\) to \((z, t)\) which moves in such a way that if \(Bf\) is applied at any time, \(s \in [0, 1]\), then \(q(s)\) will define a path from the image back to \((Y_0, 1)\). More precisely, the homotopy between the two ways of mapping \((Bf)(Y, t)\) to \(hofib(Y_0, 1)\) is going to be

\[
H : (Bf)(Y, t) \times I \rightarrow hofib(Y_0, 1)(Bf), \\
H_s(z) = (\tilde{q}(s), \varphi(s)),
\]

such that at every \(s \in [0, 1]\) it is the case that \(H_s(z)\) is in \(hofib(Y_0, 1)(Bf)\). So if \(s \in [0, 1]\) then \(\tilde{q}(s)\) needs to be a path from \(Bf(\varphi(s))\) to \((Y_0, 1)\). Define \(\varphi\) similarly to \(q\), break it up into \(n\) parts \(\varphi_i\). Recalling that

\[
t = (t^0, \ldots, t^n) \in \Delta^n, \text{ and} \\
t'_i = (t^0_i, \ldots, t^n_i) \in t^i \Delta^i,
\]

each part, \(\varphi_i(s)\), is a path from

\[
((*, g^0_1, \ldots, g_{i-1}^0, \eta_i, g^1_i, \ldots, g_{i+1}^i, *)), \\
(t^0, \ldots, t^{i-1}, t^i_0, \ldots, t^i_{l_i-1}, t^i_{l_i} + t^{i+1} + \cdots + t^n))
\]

to

\[
((*, g^0_1, \ldots, g_{i-1}^0, \eta_i, g^1_i, \ldots, g_{i+1}^i, *)), \\
(t^0, \ldots, t^{i-1}, t^i_0, \ldots, t^i_{l_i-1}, t^i_{l_i} + t^{i+1} + \cdots + t^n, 0, 0, \ldots, 0))
\]

\sim ((*, g^0_1, \ldots, g_{i-1}^0, \eta_i, g^1_i, \ldots, g_{i+1}^i, *)), \\
(t^0, \ldots, t^{i-1}, t^i_0, \ldots, t^i_{l_i-1}, t^i_{l_i} + t^{i+1} + \cdots + t^n))
and is defined to be

\[((*, g_0^i, \ldots, g_{i_0}^i, \eta_1, \ldots, \eta_i, g_1^i, \ldots, g_{i_1}^i, *),
(t^0, \ldots, t^{i-2},
(1-s) t_i^0, \ldots, \tau_{i-1}^{i-1}, (1-s) (t_i^1 + t_i^{i+1} + \cdots + t^n)\right)\).}

One can check that at any \(s \in [0, 1]\) that \(Bf(\varphi_i(s)) = q_i(s)\):

\[Bf(\varphi_i(s)) = ((*, f(g_0^i), \ldots, f(g_{i_0}^i), f(\eta_1), \ldots, f(\eta_i), f(g_1^i), \ldots, f(g_{i_1}^i), *),
(t^0, \ldots, t^{i-2},
(1-s) t_i^0, \ldots, (1-s) (t_i^1 + t_i^{i+1} + \cdots + t^n)))) = ((*, 1, \ldots, 1, h_1, \ldots, h_i, 1, \ldots, 1, *),
(t^0, \ldots, t^{i-2},
(1-s) t_i^0, \ldots, (1-s) (t_i^1 + t_i^{i+1} + \cdots + t^n))))
\sim ((*, h_1, \ldots, h_i, *),
(t^0, \ldots, t^{i-2},
(1-s) (t_0^i + \cdots + t_{i-1}^i + t_i^1 + t_i^{i+1} + \cdots + t^n))))
\]

Putting these \(\varphi_i\) together in the same way as was done with the \(q_i\) before, see that

\[Bf(\varphi(s)) = q(s).\]

Using the path \(q\) to define a path of paths, for each \(s \in [0, 1]\) one obtains

\[\tilde{q}(s) : [0, 1] \to BC',
\alpha \mapsto q(s + (1 - s) \times \alpha),\]
is a path from \( q(s) = Hf(\varphi(s)) \) to \((Y_0, 1)\). Then,

\[
H_s(z) = (\tilde{q}(s), \varphi(s)),
\]

is a homotopy with

\[
H_0(z) = (q, z),
\]

\[
H_1(z) = (c_{(Y_0, 1)}, \tau(z)),
\]

as required. \(\square\)

### 4.4 Example: A Quasi-Fibration on Configurations on Orbifolds

In this section the work from the previous sections will be used in order to get a result about quasi-fibrations on certain relative configuration spaces on orbifolds. The proof of this result will closely follow the ones found in [EVW15, McD75, Sal01].

This work looks at relative configuration spaces on orbifolds. Let \( \mathcal{M} \) be an orbifold and let \( \mathcal{N} \subseteq \mathcal{M} \) be a full sub-orbifold. Define the unordered configuration space of \( \mathcal{M} \) relative to \( \mathcal{N} \), written \( \text{Conf}(\mathcal{M}, \mathcal{N}) \), to be the orbifold with

- object space

\[
\bigsqcup_{n \geq 0} \left\{ \underline{x} \in \text{Conf}_n M_0 \mid \text{if } i \neq j \text{ then there is no arrow } x_i \to x_j \right\} / \sim,
\]

where \( \text{Conf}_n M_0 \) is the usual unordered configuration space on the object space manifold \( M_0 \) and \( \underline{x}^1 \sim \underline{x}^2 \) if \( \underline{x}^1 \cap (M_0 \setminus N_0) = \underline{x}^2 \cap (M_0 \setminus N_0) \); and

- morphism space

\[
\text{Hom}(\underline{x}^1), \underline{x}^2) \cong \text{Hom}(\underline{x}^1 \cap (M_0 \setminus N_0), \underline{x}^2 \cap (M_0 \setminus N_0)),
\]

where the left-hand-side is arrows in \( \text{Conf}(\mathcal{M}, \mathcal{N}) \) and the right-hand-side is arrows in \( \text{Conf}_s(\mathcal{M} \setminus \mathcal{N}) \), with \( s = |\underline{x}^1 \cap (M_0 \setminus N_0)| = |\underline{x}^2 \cap (M_0 \setminus N_0)| \).

Intuitively one should think of \( \text{Conf}(\mathcal{M}, \mathcal{N}) \) as being the orbifold of configurations such that points can be created or destroyed in \( \mathcal{N} \). Two configurations are connected by an arrow if the points away from \( \mathcal{N} \) are connected by arrows.

The goal of this section is the proof of the following theorem.
Theorem 4.4.1. Let \( \mathcal{M} \) be a connected \( n \)-orbifold with \(|\mathcal{M}|\) compact. Let \( \mathcal{M}' \subseteq \mathcal{M} \) be a full \( n \)-suborbifold with \(|\mathcal{M}'|\) compact, and let \( \mathcal{N} \) be a full, closed \( n \)-suborbifold such that \((\mathcal{M}', \mathcal{N} \cap \mathcal{M}')\) is connected. Then there is a quasi-fibration sequence

\[
B\text{Conf}(\mathcal{M}', \mathcal{N} \cap \mathcal{M}') \to B\text{Conf}(\mathcal{M}, \mathcal{N}) \xrightarrow{Bf} B\text{Conf}(\mathcal{M}, \mathcal{M}' \cup \mathcal{N}).
\]

To show this theorem, the work from the previous sections will be applied here in the case when, \( \mathcal{C} := \text{Conf}(\mathcal{M}, \mathcal{N}) \), \( \mathcal{C}' := \text{Conf}(\mathcal{M}, \mathcal{M}' \cup \mathcal{N}) \), and the map \( f : \mathcal{C} \to \mathcal{C}' \) is the map which increases the size of the quotient on the object space and similarly for the morphisms.

The first thing to do is to check that the map

\[
Bf : B\text{Conf}(\mathcal{M}, \mathcal{N}) \to B\text{Conf}(\mathcal{M}, \mathcal{M}' \cup \mathcal{N}),
\]

satisfies the assumptions needed for Theorem 4.3.1. The things to check in order to do this are:

1. the map

\[
(Bf)^{-1}(C'_0) \xrightarrow{Bf} C'_0
\]

is a quasi-fibration; and

2. if \( h : Z_0 \leftarrow Z_1 \) is an arrow in \( C' \) then there exists a continuous splitting of the target map \( t : E \to H^0_0 \), where

- \( E \subseteq C_1 \) is the subspace \( \{g \mid f(g) = h\} \); and

- \( H^0_0 \subseteq C_0 \) is the subspace \( \{X \mid f(X) = Z_0\} \).

Begin with showing the second condition.

Take an arrow \( h : Z_0 \leftarrow Z_1 \) in \( C' \). This means that \( Z_0 \) and \( Z_1 \) are made up of points in \( M_0 \setminus (M'_0 \cup N_0) \) and \( h \) is an arrow between the \( Z_i \) (made up of arrows in \( \mathcal{M} \)). The task is to show that there is a continuous splitting of the target map \( t : E \to H^0_0 \).

The cornerstone of this proof is the definition of \( \iota : H^0_0 \to E \), which follows. Recall that \( H^0_0 = \{X \mid f(X) = Z_0\} \), this means that for any \( X \in H^0_0 \), it is the case that \( X \cap (M_0 \setminus (M'_0 \cup N_0)) = Z_0 \). Break up \( X \) into the \( \mathcal{M}' \) part and the 'away from \( \mathcal{M}' \) part. Let \( X' := X \cap M'_0 \), then observe that \( X' \cup Z_0 = X \) and
$X' \cap Z_0 = \emptyset$. Define $\iota(X)$ to be the arrow to $X = X' \cup Z_0$ from $X' \cup Z_1$ in $C_1$ which acts by $h$ on the $Z_0 \leftarrow Z_1$ part and acts as the identity on the $X' \leftarrow X'$ part. Then the map

$$\iota : H^0_0 \to E,$$

$$X \mapsto \iota(X),$$

is a continuous splitting of the target map $t : E \to H^0_0$. One can see the continuity by looking at open neighbourhoods in $E$. Every arrow in $E$ is $h$ on the $M_n(M'_0[N])$ part, so every open neighbourhood in $E$ will not change the action on the $M \setminus (M' \cup N)$ part. A neighbourhood of a point, $g \in E$ will correspond directly to an open neighbourhood of identity arrows in the $M'_0$ part. In turn, this neighbourhood, under the pre-image of $\iota$, corresponds to an open neighbourhood of points in $M'_0$ which corresponds to an open neighbourhood in $H^0_0$. Thus $\iota$ is continuous.

Now that the continuous splitting has been constructed, Theorem 4.4.3 will be proved if the following lemma can be shown.

**Lemma 4.4.2.** The map,

$$(Bf)^{-1}(C'_0) \xrightarrow{Bf} C'_0,$$

is a quasi-fibration.

**Proof.** The proof will use the standard Dold-Thom criterion for quasi-fibrations. To use this criterion, filter $C'_0$, and its pre-image, on the number of points in the configuration away from $M'_0$. To filter $C'_0$, let $G_k$ be the subspace of $C'_0$ with at most $k$ points in $M'_0$. Then it is the case that

$$T_k := G_k \setminus G_{k-1} \cong \text{obj}(\text{Conf}_k(M \setminus (M' \cup N))),$$

that is, all the configurations with exactly $k$ points in $M \setminus (M' \cup N)$.

In this case, the Dold-Thom criterion for quasi-fibrations is used on the map over the object space, the version presented here is as stated in [11].

**Theorem 4.4.3.** Assume that,

1. $Bf : (Bf)^{-1}(T_k) \to T_k$ is a fibration, with fibre $B\text{Conf}(M',N \cap M')$;
2. there is an open subset $U_k \subseteq G_{k+1}$, with $G_k \subseteq U_k$ and homotopies $h_t : G_k \to G_k$ and $H_t : (Bf)^{-1}(G_k) \to (Bf)^{-1}(G_k)$ such that
   (a) $h_0 = \text{id}$, $h_t(G_k) \subseteq G_k$ and $h_t(U_k) \subseteq G_k$;
   (b) $H_0 = \text{id}$, $(Bf) \circ H_t = h_t \circ (Bf)$; and
4.4. EXAMPLE: CONFIGURATIONS ON ORBIFOLDS

(c) $H_1 : (Bf)^{-1}(\mathfrak{x}) \to (Bf)^{-1}(h_1(\mathfrak{x}))$ is a homotopy equivalence for each $\mathfrak{x} \in U_k$.

Then the map

$$(Bf)^{-1}(C'_0) \xrightarrow{Bf} C'_0,$$

is a quasi-fibration with fibre $BConf(M', N \cap M')$.

Similar to the filtration of $C'_0$, the space $(Bf)^{-1}(C'_0)$ is filtered analogously. Firstly, note that $(Bf)^{-1}(C'_0) = BD$, where $D$ is the category with

- $\text{obj}D = \text{obj}C$; and

- $\text{mor}D = \{g \in \text{mor}C | g$ is the identity on $M \setminus M'\}$.

As with all classifying spaces, a point in $BD$ is defined by a simplex $Y_0 \leftarrow \cdots \leftarrow Y_n$ and a point $t \in \Delta^n$. To filter $BD$, define $F_k$ to be the subspace of $BD$ made of points $(Y_0 \leftarrow \cdots \leftarrow Y_n, t)$ such that each $Y_i$ has at most $k$ points in $M \setminus (M' \cup N)$. Note that because $M'$ and $N$ were defined to be full sub-orbifolds, all arrows are internal, so each $Y_i$ will have the same number of points in $M \setminus (M' \cup N)$.

**Claim 4.4.4.** Let $S_k := F_k \setminus F_{k-1}$. The spaces $S_k$ and $T_k$ satisfy the relation,

$$S_k := F_k \setminus F_{k-1} \cong \text{obj}(Conf_k(M \setminus M', N \setminus (M' \cap N))) \times BConf(M', M' \cap N),$$

$$= T_k \times BConf(M', M' \cap N).$$

**Proof of Claim.** For $\mathfrak{x} \in \text{obj}(Conf_k(M \setminus M', N \setminus (M' \cap N)))$, define a continuous functor

$$\rho_\mathfrak{x} : Conf(M', N \cap M') \to Conf(M, N),$$

$$y \mapsto \mathfrak{x} \cup y, \text{ on objects, and}$$

$$\gamma \mapsto id_{\mathfrak{x}} \cup \gamma, \text{ on morphisms.}$$

The map

$$\text{obj}(Conf_k(M \setminus M', N \setminus (M' \cap N)))$$

$$\to \text{Funct}^{\text{str}}(Conf(M', N \cap M') \to Conf(M, N),$$

$$\mathfrak{x} \mapsto \rho_\mathfrak{x},$$
CHAPTER 4. A QUASI-FIBRATION CRITERION

is continuous. This induces a continuous map

$$\text{obj}(\text{Conf}_k(M \setminus M', N \setminus (M' \cap N)))$$

$$\rightarrow \text{Map}(\text{BConf}(M', N \cap M') \rightarrow \text{BConf}(M, N)),$$

where $\rho^*_\mathcal{Z}$ is the map

$$\rho^*_\mathcal{Z} : \text{BConf}(M', N \cap M') \rightarrow \text{BConf}(M, N),$$

$$(Y_0 \leftarrow \cdots \leftarrow Y_n, t) \mapsto (\mathcal{Z} \cup Y_0 \leftarrow \cdots \leftarrow \mathcal{Z} \cup Y_n, t).$$

The arrows on the right hand side are the obvious choice. That is, they are
the same arrows as on the left in the $M'$ part and the identity on the $\mathcal{Z}$ part,
the part away from $M'$. Note that since the arrows on the right hand side are
the identity away from $M'$, the image of $\rho^*_\mathcal{Z}$ lies in $S_k$. (Recall that $S_k$ is the
subspace of $BD$ with exactly $k$ points away from $M'$, and all arrows in $D$ are
the identity away from $M'$.)

One can also define a map back from $S_k$, say $\gamma$. Start with a point $(Z_0 \leftarrow \cdots \leftarrow Z_n, t)$ in $S_k$. Since every arrow in $D$ is the identity away from $M'$, it is
possible to define $\mathcal{Z} = Z_0 \cap (M_0 \setminus M'_0)$ and have it satisfy $\mathcal{Z} = Z_i \cap (M_0 \setminus M'_0)$
for every $i$. The separation map is defined to be,

$$\gamma : S_k \rightarrow \text{obj}(\text{Conf}_k(M \setminus M', N \setminus (M' \cap N))) \times \text{BConf}(M', M' \cap N),$$

$$\gamma(Z_0 \leftarrow \cdots \leftarrow Z_n, t) = (\mathcal{Z}, (Z_0 \leftarrow \cdots \leftarrow Z_n \setminus \mathcal{Z}, t)).$$

Similar to above, the arrows on the right are just the arrows on the left restricted
to the $M'$ part. Both of the compositions are now the identity, $\rho^*_\mathcal{Z} \circ \gamma = id$ and
$\gamma \circ \rho^*_\mathcal{Z} = id$. Thus this is a homeomorphism

$$S_k \cong \text{obj}(\text{Conf}_k(M \setminus M', N \setminus (M' \cap N))) \times \text{BConf}(M', M' \cap N).$$

Using these filtrations of the spaces, observe that for every $k$ there a pre-
image homeomorphism

$$\alpha_k : (Bf)^{-1}(T_k) \cong T_k \times \text{BConf}(M', N \cap M'),$$

so that $Bf \circ \alpha_k^{-1}$ is the projection onto the factor $T_k$.

The following sentence will be explained in the subsequent sentences. Take
a collared neighbourhood $\mathcal{U}$ of $M'$ in $M$ and a smooth isotopy retraction $r :
\( U \times I \to \mathcal{M} \) with \( r_1(U \cap \mathcal{N}) \subseteq \mathcal{N} \). A collared neighbourhood \( U \) of \( \mathcal{M} \) in \( \mathcal{M} \) is a full sub-orbifold \( U \subseteq \mathcal{M} \) so that \( U_0 \) is a collared neighbourhood of \( M'_0 \) in \( M_0 \). A smooth isotopy retraction \( r : U \times I \to \mathcal{M} \) is a smooth isotopy retraction (on manifolds) \( U_0 \times I \to M'_0 \) which commutes with the actions of the arrow spaces. Also note that when it is said that \( r : U \times I \to \mathcal{M} \) is a smooth isotopy retraction, what is really meant is that \( r \) is a map \( r : U \times I \to \mathcal{M} \) such that \( r_1(U) \subseteq \mathcal{M} \).

For each \( k \), there is an open neighbourhood \( U^k \) of \( G_k \) in \( G_{k+1} \) (think a configuration in \( G_{k+1} \) with a point really close to \( \mathcal{M}' \)) so that \( r \) induces a smooth isotopy retraction \( r^k : U^k \times I \to G_k \) and a smooth isotopy retraction \( \tilde{r}^k : (Bf)^{-1}(U^k) \times I \to (Bf)^{-1}(G_k) \) covering \( r^k \).

To show that the map is a quasi-fibration, it needs to be the case that that if \( P \in U^k \), then the restriction \( t : (Bf)^{-1}(P) \to (Bf)^{-1}(r^k(P)) \) of \( \tilde{r}^k \) is a weak homotopy equivalence. Use \( \alpha_k \) to identify the domain and range of \( t \) with \( BConf(\mathcal{M}', \mathcal{M}' \cap \mathcal{N}) \). Then \( t \) pushes points away from the boundary of \( \mathcal{M}' \) with \( \mathcal{M} \), then adds a finite set of points in proximity to this boundary. Using the assumption that \( (\mathcal{M}', \mathcal{M}' \cap \mathcal{N}) \) is connected, one can then push the points that were just added into \( \mathcal{N} \), which allows them to be forgotten from the configuration. Since all the points that were added have now been forgotten, the map \( t \) has, in effect, done nothing. Thus, the map \( t \) is homotopic to a homeomorphism.

This concludes checking all of the conditions needed to invoke the Dold-Thom Criterion for quasi-fibrations. The map,

\[
(Bf)^{-1}(C_0') \xrightarrow{Bf} C_0',
\]

is a quasi-fibration. \(\square\)

With the completion of this proof, both of the assumptions needed for Theorem 4.3.1 to be true in this case are satisfied. One can therefore conclude that the map

\[
BConf(\mathcal{M}', \mathcal{N} \cap \mathcal{M}') \to BConf(\mathcal{M}, \mathcal{N}) \xrightarrow{Bf} BConf(\mathcal{M}, \mathcal{M}' \cup \mathcal{N}),
\]

is a quasi-fibration sequence.
Chapter 5

Relation to Mapping Spaces of Orbifolds

The goal of this chapter is to explain how it is expected that the quasi-fibration work from the previous chapter can be used. In the case of manifolds, a standard technique is to show that certain configuration spaces are homotopic to mapping spaces. The reason for doing this is so that one can gather important information about the homology of configurations from mapping spaces, and vice-versa.

5.1 Already Known Results for Manifolds

This section will talk about previous work on manifolds, references include [McD75, EVW15, Sal01].

Definition 5.1.1. Let $X$ be a manifold. For $x \in X$, let $D_x$ denote the unit disc in the tangent space of $X$ at $x$, $T(X)_x$, and let $S_x := D_x/\partial D_x$. Define,

- $E_X$ to be the bundle associated to $T(X)$ such that the fibre at $x$ is $S_x$ ($S_x$ has base-point $*_x$, the image of $\partial D_x$);

- $\Gamma(N)$ to be the space of continuous sections of $E_X$ over $N$, where $N \subseteq X$; and

- $\Gamma(N, L')$ to be the subspace of $\Gamma(N)$ such that the sections over every $x \in L'$ equal $*_x$, where $L' \subseteq N$.

It may also be of use to note that $E_X$ defined above is homeomorphic to the fibrewise one-point compactification of $T(X)$. 

105
The standard map which is used in these theorems is the scanning map,

$$Conf(X, \partial X) \to \Gamma(X),$$

the general idea of which takes a configuration $z \in Conf(X, \partial X)$ and gives a section which maps $x \in X$ to the restriction of $z$ to a small disc neighbourhood of $x$, modulo its boundary.

**Theorem 5.1.2.** [McD75, Theorem 2.6] Let $X$ be a manifold with boundary $\partial X = L \cup L'$ such that $\dim(L) = \dim(L') = \dim(\partial X)$ and $\partial L = \partial L' = L \cap L'$. Then if $X$ is connected and $L$ is non-empty, there is a homotopy equivalence,

$$Conf(X, L) \to \Gamma(X, L').$$

For the detailed proof of this theorem, refer to [McD75] or [Sal01]. The remainder of this section will just touch on why the work from Chapter 4 is important.

The proof of Theorem 5.1.2 is an inductive argument. The base case for the induction involves setting

$$M := J^k \times I^{n-k},$$
$$N := (DJ^{k-1} \times J \times I^{n-k}) \cup (J^{k-1} \times [-1,0] \times I^{n-k}),$$
$$M' := J^{k-1} \times [1,2] \times I^{n-k},$$

where $I := [0,1], J := [-1,2]$ and $DJ := [-1,0] \cup [1,2]$. Then put this data into the diagram,

$$\begin{array}{ccc}
Conf(M', N \cap M') & \longrightarrow & Conf(M, N) \\
\downarrow & & \downarrow \cong \\
\Omega Conf(M, N \cup M') & \longrightarrow & Path(Conf(M, N \cup M')) \\
\end{array}$$

The top row of this diagram is a quasi-fibration (the manifold analogue of the main result of Chapter 4), and the bottom row of this diagram is the path-loop fibration. The right vertical arrow is the identity arrow (therefore a homotopy equivalence), and the middle vertical arrow is a homotopy equivalence because both spaces are contractable. Therefore the induced map, the left vertical arrow $Conf(M', N \cap M') \to \Omega Conf(M, N \cup M')$, is a homotopy equivalence. A picture of the top row for $n = 2$ and $k = 2$ can be seen in Figure 5.1.

The next part of the proof is to show that $Conf(X, L) \to \Gamma(X, L')$ is an equivalence when $X = M$ and $L = N \cup M'$. This is done by showing inductively that $\Omega Conf(M, N \cup M')$ is homotopy equivalent to $\Gamma(X, L')$. For the base case,
5.1. ALREADY KNOWN RESULTS FOR MANIFOLDS

Figure 5.1: A quasi-fibration sequence on relative configuration spaces of manifolds. The corresponding quasi-fibration sequence is $\text{Conf}(M', M' \cap N) \rightarrow \text{Conf}(M, N) \rightarrow \text{Conf}(M, N \cup M')$.

set $L = N \cup M'$ to be the entire boundary of $M$ ($k = n$) (See Figure 5.1 for $n = 2$). Then any configuration on in $\text{Conf}(M, N \cup M')$ can be radially expanded so that there is at-most one point left in the configuration (the other points are dropped when they get to $L$). This is homotopic to $(M, N \cup M') \simeq S^n$.

As for the section space $(M; \emptyset)$, observe, $(M; \emptyset) \simeq (\emptyset; \emptyset) \simeq D^n = \partial D^n \simeq S^{n-1}$.

The inductive step, assume that $\text{Conf}(X, L) \rightarrow \Gamma(X, L')$ is a homotopy equivalence for $k > k'$ and show the equivalence for $k = k'$. Let $\partial_k = \partial I^k \times I^{n-k} \subseteq I^n$. The section space $\Gamma(I^n, \partial_k) = \text{Map}((I^n, \partial_k), (S^n, *))$ is homotopy equivalent to $\text{Map}(I^k, \partial I^k), (S^n, *))$ by collapsing the $I^{n-k}$ part to a point. Then $\text{Map}(I^k, \partial I^k), (S^n, *)) = \Omega^k S^n$, so the quasi-fibration sequence and path-fibration sequence diagram above give the inductive step,

$$\text{Conf}(M', N \cap M') = \Omega \text{Conf}(M, N \cup M'),$$

$$= \Omega \Gamma(I^n, \partial_k),$$

$$= \Omega^k S^n,$$

$$= \Omega^{k+1} S^n,$$

$$= \Gamma(I^n, \partial_{k+1}),$$

as required.

Having this work, the argument to get the entire theorem iteratively attaches handles of $X$. Suppose $M'$ is obtained by attaching a handle of index $i$, say $H$, then...
to $M''$. Then the sequences,

$$
\begin{align*}
\text{Conf}(H, \partial H \setminus (\partial H \cap M'')) & \rightarrow \text{Conf}(M', \partial M') \rightarrow \text{Conf}(M'', \partial M'') \\
\cong & \rightarrow \\
\Gamma(H/(H \cap M'')) & \rightarrow \Gamma(M') \rightarrow \Gamma(M''),
\end{align*}
$$

where the top sequence is a quasi-fibration and the bottom sequence is a fibration, form the inductive step in the proof by the five-lemma. See [Sal01] for a more rigorous proof.

5.2 Analogous Ideas on Orbifolds

Moving into the world of orbifolds, using techniques from the previous section are now more difficult. It was noted that the main result of Chapter 4 will be of great use, giving a quasi-fibration sequence

$$
B\text{Conf}(\mathcal{M}', \mathcal{N} \cap \mathcal{M}') \rightarrow B\text{Conf}(\mathcal{M}, \mathcal{N}) \xrightarrow{BJ} B\text{Conf}(\mathcal{M}, \mathcal{M}' \cup \mathcal{N}),
$$

for $\mathcal{M}, \mathcal{N}$ and $\mathcal{M}'$ certain orbifolds. What is not clear, however, is what the orbifold analogue for the spaces $\Gamma(\mathcal{N}, L')$ should be.

Returning to the manifold setting for a moment, there is an alternate way to view the spaces $\Gamma(\mathcal{N}, L')$. The definition above relied on subspaces $L' \subseteq \mathcal{N} \subseteq X$ and a bundle $E_X$ associated to $T(X)$ such that the fibre at $x$ is $S_x$. The section space $\Gamma(\mathcal{N}, L')$ was then defined as the space of continuous sections of $E_X$ over $\mathcal{N}$ such that the sections over every $x \in L'$ equal $*_x$.

Another way of viewing the space $\Gamma(\mathcal{N}, L')$ is by defining a manifold, $T_X$, and using it in place of $E_X$. Define,

$$
T_X := \{(x, z) \mid x \in X, z \in \text{Conf}(\overline{B}_\varepsilon(x), \partial)\},
$$

which consists of points $x \in X$ with configurations in a closed $\varepsilon$-ball around $x$, where points in the configuration can be added or removed on the boundary of the $\varepsilon$-ball. This space $T_X$ lies over $X$, the fibre at $x \in X$, is the space of configurations in the closed $\varepsilon$-ball around $x$, the base-point of the fibre at $x$ is $(x, \emptyset), x$ with the empty configuration. Note that for a fixed $x \in X$, a configuration in an $\varepsilon$-ball around $x$ can be radially expanded (as done in the previous section), leaving at most one point in the configuration (the other points were pushed onto the boundary of the $\varepsilon$-ball and were dropped). The configuration space of at most one point in the closed $\varepsilon$-ball, relative to the boundary, is homotopic to $S^n$, where $n$ is the dimension of $X$. Therefore the


fibre over \(x\) in \(T_X\) is homotopic to the fibre over \(x\) in \(E_X\).

Unfortunately, it is currently unknown to the author exactly how an analogous section space in the world of orbifolds should be defined. What follows will be ideas for future definitions.

A potential way to define \(T_X\) is the following. This process involves taking \(\varepsilon\)-balls around points in \(X_0\), which relies on a choice of metric. Any metric will do, so one can simply assume that a metric exists. Note that this assumption isn’t one that is particularly wanted, but it is needed (and probably adds more restrictions on \(X_0\), for example smoothness).

**Definition 5.2.1.** Let \(\mathcal{X}\) be an orbifold such that a metric can be defined on \(X_0\). Let \(\varepsilon \in \mathbb{R}_{>0}\) (a good choice of \(\varepsilon\) is half of the injectivity radius of the map described in Chapter 3 as measured by the metric on \(X_0\)). Define \(T_X\) to be the orbifold with

- object space

\[
\{(x, \tilde{z}) \mid x \in \text{obj}(\mathcal{X}), \tilde{z} \in \text{obj}(\text{Conf}(\overline{B}_\varepsilon(x), \partial))\},
\]

where \(\overline{B}_\varepsilon^*(x)\) is the smallest full sub-orbifold of \(\mathcal{X}\) which contains the closed \(\varepsilon\)-ball around \(x\) in \(X_0\) (see Definition 3.1.1 for full sub-orbifold); and

- arrows from \((x_1, \tilde{z}_1)\) to \((x_2, \tilde{z}_2)\) being elements of \(X_1 \times \text{mor}(\text{Conf}(\overline{B}_\varepsilon^*(x_1), \partial))\).

Note that this arrow space is well defined since

a. if there is no arrow \(x_1 \rightarrow x_2\) in \(X_1\) then there is no arrow \((x_1, \tilde{z}_1) \rightarrow (x_2, \tilde{z}_2)\) in \(T_X\); and

b. if there does exist an arrow \(x_1 \rightarrow x_2\), then \(\overline{B}_\varepsilon^*(x_1) = \overline{B}_\varepsilon^*(x_2)\) and so

\[
\text{mor}(\text{Conf}(\overline{B}_\varepsilon^*(x_1), \partial)) = \text{mor}(\text{Conf}(\overline{B}_\varepsilon^*(x_2), \partial)),
\]

which justifies why \(\text{mor}(\text{Conf}(\overline{B}_\varepsilon^*(x_1), \partial))\) is an acceptable choice in the arrow space of \(T_X\).

This construction is independent (up to homotopy equivalence) of the choice of \(\varepsilon\).

One should note that the ball \(\overline{B}_\varepsilon^*(x)\) could be made up of several disjoint \(\varepsilon\)-balls in \(X_0\) (when \(\varepsilon\) is small) or it is possible that they could overlap (See Figure 5.2). The points in the configurations are only allowed to be added or removed on the boundary of the entire \(\overline{B}_\varepsilon^*(x)\) space. This is a big contributor to the difficulty in using the argument for manifolds in the orbifold setting, as radial expansion of configuration points away from \(x\) is no-longer possible.
Figure 5.2: A illustration of $\overline{B}_\varepsilon(x)$ in the orbifold $[D^2/\mathbb{Z}_6]$. Note that the orbits of $\overline{B}_\varepsilon(x)$ overlap, leading to this ‘bouquet’ picture. Points can only be added or removed from configurations on the edges of the entire orbit. A smaller value of $\varepsilon$ would lead to disjoint $\varepsilon$-balls. (Though in this case moving $x$ closer to the origin would eventually make the $\varepsilon$-balls join up again).

As for the section space, $\Gamma(\mathcal{X})$ is a category of orbifold maps $\mathcal{X} \to T_X$ which descend back down to the identity map. Such maps consist of a continuous map on the objects, and a continuous map on the arrows. The map on objects is of the form,

$$X_0 \to \text{obj}(T_X),$$

$$x \mapsto (x, z),$$

where each $z \in \text{obj}(\text{Conf}(\overline{B}_\varepsilon(x), \partial))$. The map on arrows should be of the form,

$$X_1 \to \text{mor}(T_X),$$

$$x \overset{g}{\to} y \mapsto (x \overset{g}{\to} y, \{z_1 \overset{\alpha_1}{\to} z'_1, \ldots, z_n \overset{\alpha_n}{\to} z'_n\}).$$

Such functors indeed descend back down to the identity functor. These functors form the object space of $\Gamma(\mathcal{X})$, the arrow space is all natural transformations between these functors.

Consider the case when $\mathcal{X} = [D^n/G]$. Such orbifolds can be thought of as the handles used to build a larger orbifold (similar to $D^n$ being a handle in the manifold setting), [Wast05, Hepo07]. It is then possible to get a homotopy
equivalence,
\[ \Gamma([D^n/G]) \simeq \Gamma([*/G]), \]
by contracting the disc in \([D^n/G]\) down to the centre. The orbifold maps then become,
\[ \begin{align*}
[*/G] \rightarrow TX, \\
* \mapsto (**, z), \\
* \xrightarrow{g} \rightarrow (**, \{z_1 \xrightarrow{\alpha_1} z_1, \ldots, z_n \xrightarrow{\alpha_n} z_n\}),
\end{align*} \]
where \( z \in \text{obj}(Conf\left(\overline{B}_c(\ast, \partial)\right)) \) and each \( \alpha_i \) is in the isotropy group at \( z_i \).

From here the hope is to use a map of the form,
\[ \text{obj}(Conf(\mathcal{X}, \mathcal{L})) \rightarrow \text{obj}(\Gamma(\mathcal{X})), \]
\[ z \mapsto \left( x \mapsto z \cap \overline{B}_c(x) \right) \]
\[ \left( g : x \rightarrow y \mapsto (g : x \rightarrow y, \{id : z_1 \rightarrow z_1, \ldots, id : z_n \rightarrow z_n\}) \right). \]
Unfortunately this map doesn’t result in a homotopy equivalence. The definitions of \( \Gamma(\mathcal{X}) \), and the map from configurations, need to be tweaked in an as-yet unknown way to find a homotopy equivalence.

This is the extent of the progress with these objects. The hope is that with the right definition of \( \Gamma(\mathcal{X}, \mathcal{L}) \) that it will be possible to prove an equivalence of orbifolds,
\[ Conf(\mathcal{X}, \mathcal{L}) \rightarrow \Gamma(\mathcal{X}, \mathcal{L}). \]
Chapter 6

The Salvetti Complex

This final chapter moves away from the general direction of the previous work and looks into some computer programming and calculations. There are no orbifolds or ghosts here, just configurations on the standard two-dimensional disc (that is the classifying space of the braid groups).

The motivation of this work was an urge to calculate the homology of Hurwitz spaces. What these spaces are is not important for the following work, only that to calculate these homologies, a topological description of the braid group which can easily be worked with is needed. One such description is the Salvetti complex, which was introduced by Salvetti in 1987 [Sal87]. This chapter will use the simplicial description of the Salvetti complex as given by Paris in [Par93], the construction of which is described in the following Section 6.1. In addition to this description, Clancy and Ellis gave an alternate, cellular description of the Salvetti complex, found in [CE10]. The point of using the simplicial description of the Salvetti complex is that it is possible, in the pure braid group case, to represent the complex in a computer program, this is the subject of Section 6.2. A full computer implementation for the pure braid group on $n$ strands can also be found in Appendix A.

6.1 Salvetti Complex Background

The Salvetti complex is a useful tool for investigating the compliment of hyperplane arrangements, which will now be introduced.

**Definition 6.1.1.** Let $V$ be a real vector space. An *arrangement of hyperplanes* in $V$, denoted $\mathcal{A}$, is a finite set of hyperplanes through the origin of $V$. 
Furthermore, the arrangement $\mathcal{A}$ is called essential if

$$\bigcap_{H \in \mathcal{A}} H = \{0\}.$$ 

For a given hyperplane arrangement $\mathcal{A}$ in $V$, define the complexified complement

$$M(\mathcal{A}) := V_C - \left( \bigcup_{H \in \mathcal{A}} H_C \right),$$

where $V_C = C \otimes V$ is the complexification of $V$, and for $H \in \mathcal{A}$, $H_C$ is the complex hyperplane of $V_C$ spanned by $H$. Note that the complexification of the hyperplanes satisfy $H_C = H + iH \subset V_C$.

The Salvetti complex corresponding to a hyperplane arrangement is a way to work with the real vector space $V$ and hyperplane arrangement $\mathcal{A}$ to get a simplicial complex which is homotopic to $M(\mathcal{A})$.

The following theorem due to Salvetti in [Sal87], though not quite using this particular notation, explains why the Salvetti complex is an object one should study.

**Theorem 6.1.2.** Let $\mathcal{A}$ be an essential hyperplane arrangement. The Salvetti complex, $\text{Sal}(\mathcal{A})$ is a simplicial complex such that

$$\text{Sal}(\mathcal{A}) \simeq M(\mathcal{A}).$$

The Salvetti complex relies on the hyperplane arrangement $\mathcal{A}$ decomposing $V$ into facets. Suppose that $\mathcal{B} \subseteq \mathcal{A}$, then the hyperplanes in $\mathcal{A} \setminus \mathcal{B}$ cut $\cap_{H \notin \mathcal{B}} H$ into facets of the arrangement $\mathcal{A}$. Note here that if $\mathcal{B} = \emptyset \subseteq \mathcal{A}$, then $\cap_{H \notin \mathcal{B}} H = V$, showing that each connected component of $V \setminus \cup_{H \notin \mathcal{A}} H$ is a facet of $\mathcal{A}$ (specifically, these are called the chambers of $\mathcal{A}$).

To be more precise, the set of facets of $\mathcal{A}$ is the set

$$\bigcup_{\mathcal{B} \subseteq \mathcal{A}} \left\{ \text{comp} \left( \left( \bigcap_{H \in \mathcal{B}} H \right) \setminus \bigcup_{H' \notin \mathcal{A} \setminus \mathcal{B}} \left( \left( \bigcap_{H \in \mathcal{B}} H \right) \bigcap H' \right) \right) \right\},$$

where $\text{comp}(X)$ is the set of components of $X$. Write $\mathcal{F}(\mathcal{A})$ for the set of facets of $\mathcal{A}$ and $|F|$ for the span of $F$ in $V$. A chamber is a co-dimension 0 facet. Define a partial order on the set of facets by defining $F \leq F'$ if $F$ and $F'$ satisfy $F \subseteq \overline{F'}$. Furthermore, say that $F < F'$ if $F \leq F'$ and $F \neq F'$.

The aim is to use the Salvetti complex to get a simplicial description of braid groups. The following theorem is well-known.
Theorem 6.1.3. The classifying space of the pure braid group on \( n \) strands is the ordered configuration space of \( n \) points in the complex plane

\[
PConf_n(\mathbb{C}) := \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid i \neq j \text{ then } x_i \neq x_j\}.
\]

One can also note that \( PConf_n(\mathbb{C}) \) is the space \( \mathbb{C}^n \) with the hyperplanes \( x_i = x_j \) removed. That is,

\[
PConf_n(\mathbb{C}) = \mathbb{C}^n \setminus \bigcup_{i \neq j} \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i = x_j\}.
\]

In the language above, \( M(A) = PConf_n(\mathbb{C}) \). Letting the real vector space \( V = \mathbb{R}^n \), with \( A \) the set of hyperplanes of the form \( \{x \in \mathbb{R}^n \mid x_i = x_j\} \) gives the correct structure for \( M(A) \). It is important to note that \( A \) as defined here is clearly not an essential hyperplane arrangement, as there is a one-dimensional subspace

\[
\{(t, t, \ldots, t) \mid t \in \mathbb{R}\} \subseteq \bigcap_{H \in A} H
\]

In order to get around this, it is possible to project out the line \( \{(t, t, \ldots, t) \mid t \in \mathbb{R}\} \) to obtain an essential hyperplane arrangement.

Build the Salvetti complex of a real vector space \( V \) of dimension \( l \) with an essential arrangement of hyperplanes \( A \). This construction will follow that of \( Par93 \). The hyperplane arrangement determines a cellular decomposition of \( S^{l-1} = \{x \in V \mid |x| = 1\} \). Pick \( F \in \mathcal{F}(A) \) such that \( F \neq \{0\} \) (\( F \) a non-zero facet). Then \( F \) corresponds to the open cell \( \Delta(F) := F \cap S^{l-1} \). Furthermore, every cell in this decomposition of \( S^{l-1} \) takes such a form.

Taking this cellular decomposition, construct a simplicial decomposition of \( S^{l-1} \). This specific simplicial decomposition is known as the ‘barycentric subdivision’. For each \( F \in \mathcal{F}(A) \) with \( F \neq \{0\} \), fix a point \( x(F) \in \Delta(F) \). Then whenever there is a chain of ascending facets

\[
\{0\} \neq F_0 < F_1 < \cdots < F_r, \quad F_i \in \mathcal{F}(A),
\]

there is a corresponding \( r \)-simplex in the barycentric subdivision, \( \varphi = x(F_0) \lor x(F_1) \lor \cdots \lor x(F_r) \), which has vertices \( x(F_0), x(F_1), \ldots, x(F_r) \). Every simplex in this simplicial decomposition is described in this way.

From this simplicial decomposition of \( S^{l-1} \), construct a simplicial decomposition of \( D^l = \{x \in V \mid ||x|| \leq 1\} \), the unit disc. This is done by taking the cone of \( S^{l-1} \). Specifically, add the vertex \( x(\{0\}) = 0 \) to the set of vertices obtained for the simplicial decomposition for \( S^{l-1} \) above. Then take any chain of ascending
facets

\[ F_0 < F_1 < \cdots < F_r, \quad F_i \in \mathcal{F}(A), \]

noting that this time it is valid to choose \( F_0 = \emptyset \). This determines an \( r \)-simplex \( \varphi = x(F_0) \vee x(F_1) \vee \cdots \vee x(F_r) \) in \( D^l \), with vertices \( x(F_0), x(F_1), \ldots, x(F_r) \). Every simplex in this simplicial decomposition of \( D^l \) is described in this way. Note that if \( F_0 \neq \emptyset \) then the simplex \( \varphi \) also appears in the simplicial decomposition of \( S^{l-1} \) presented above.

**Definition 6.1.4.** Let \( A \) be an essential hyperplane arrangement. Define the lattice of \( A \) to be the set

\[ \mathcal{L}(A) = \left\{ \bigcap_{H \in B} H \bigm| B \subseteq A \right\}. \]

For \( X \in \mathcal{L}(A) \), define

\[ A_X = \{ H \in A \mid H \supseteq X \}, \]

that is, all the hyperplanes in \( A \) which contain all of \( X \).

**Definition 6.1.5.** Let \( A \) be an essential hyperplane arrangement, \( F \in \mathcal{F}(A) \) and \( C \) a chamber of \( A \). Define \( C_F \) to be the unique chamber of \( A_{|F|} \) which contains \( C \).

**Definition 6.1.6 (Salvetti Complex).** For \( X \in \mathcal{L}(A) \) and \( D \) a chamber of \( A_X \), fix a point \( y(D) \in D \). For \( F \in \mathcal{F}(A) \) and \( C \) a chamber of \( A \), set

\[ z(F, C) := x(F) + iy(C_F). \]

The point \( z(F, C) \in V_C = C \otimes V \), since both \( x(F) \) and \( y(C_F) \) are elements of \( V \). Also note that \( z(F_1, C_1) = z(F_2, C_2) \) if and only if both \( F_1 = F_2 \) and \( (C_1)_{F_1} = (C_2)_{F_2} \).

Let \( V(Sal) \) (the set of vertices/0-simplices of \( Sal \)) be an abstract set such that

\[ \psi : \{ z(F, C) \mid F \in \mathcal{F}(A), C \text{ a chamber of } A \} \rightarrow V(Sal) \]

is a bijection. For \( F \in \mathcal{F}(A) \) and \( C \) a chamber of \( A \), let

\[ w(F, C) := \psi(z(F, C)). \]

If \( F_1, F_2 \in \mathcal{F}(A) \) and \( C \) is a chamber of \( A \) then define that \( w(F_1, C) < \)
w(F_2, C) if F_1 < F_2. Theorem 3.1 of \cite{Par93} proves that this is a partial order on V(Sal).

An r-simplex of Sal(A) is \( \Phi = w_0 \lor w_1 \lor \cdots \lor w_r \), where \( w_0 < w_1 < \cdots < w_r \) is an \((r + 1)\)-chain of ascending elements of V(Sal).

Note that in this definition, the vertices of the Salvetti complex are defined to be an abstract set in bijection with the \( z(F, C) \). To avoid extra complexities, this distinction between V(Sal) and the set of \( z(F, C) \) will become blurred in the sequel.

Example 6.1.7 \((V = \mathbb{R}, \mathcal{A} = \{\{0\}\})\). Suppose \( V = \mathbb{R} \) and let \( \mathcal{A} = \{\{0\}\} \). Begin by looking at \( M(\mathcal{A}) \), the parts that are used to build \( M(\mathcal{A}) \) are \( V_C \), which will simply be the complex plane, and \( \{0\}_C \), is just a point. Putting this together gives \( M(\mathcal{A}) = C \setminus \{0\} \simeq S^1 \), so it should be expected that the Salvetti complex in this case is homotopic to \( S^1 \).

To build the Salvetti complex for \( \mathcal{A} \), first use \( \mathcal{A} \) to obtain a cellular decomposition of \( S^0 \). This cellular decomposition simply cuts \( S^0 \) into two parts, \( \{1\} \) and \( \{-1\} \).

Recall the definition of the facets of \( \mathcal{A} \),

\[
\bigcup_{\mathcal{B} \subseteq \mathcal{A}} \left\{ \text{comp} \left( \left( \bigcap_{H \in \mathcal{B}} H \right) \setminus \bigcup_{H' \in \mathcal{A} \setminus \mathcal{B}} \left( \left( \bigcap_{H \in \mathcal{B}} H \right) \bigcap H' \right) \right) \right\}.
\]

The two possible values of \( \mathcal{B} \subseteq \mathcal{A} \) are \( \emptyset \) and \( \{\{0\}\} \). When \( \mathcal{B} = \emptyset \), the equation gives the components of the space,

\[
\mathbb{R} \setminus \{0\},
\]

which are \( \mathbb{R}_{<0} \) and \( \mathbb{R}_{>0} \). When \( \mathcal{B} = \{\{0\}\} \), the equation gives the components of the space,

\[
\{0\} \setminus \emptyset,
\]

which is just \( \{0\} \). Therefore, the set of facets of \( \mathcal{A} \) is \( \mathcal{F} = \{\{0\}, \mathbb{R}_{<0}, \mathbb{R}_{>0}\} \). Recalling the definition of \( F \leq F' \) when \( F \subseteq F' \), the partial order on the facets is completely described by,

\[
\{0\} < \mathbb{R}_{<0},
\]

\[
\{0\} < \mathbb{R}_{>0}.
\]
CHAPTER 6. THE SALVETTI COMPLEX

Pick the \( x(F) \) points for \( F \in \mathcal{F}, F \neq \{0\} \),

\[
x(\mathbb{R}_{>0}) = 1, \quad \text{and} \quad x(\mathbb{R}_{<0}) = -1.
\]

Note that there are only two points in the barycentric subdivision of \( S^0 \) here, as the only two ascending chains are \( \mathbb{R}_{<0} \) and \( \mathbb{R}_{>0} \).

From this simplicial complex for \( S^0 \), produce a simplicial decomposition of \( D^1 \) by adding in the centre point \( x(\{0\}) = 0 \). The new facet representatives for this setting are,

\[
\{-1\}, \quad \{0\}, \quad \{1\},
\]

and the two ascending chains of facets, which will give the 1-simplices of \( \text{Sal}(V) \) are,

\[
\{0\} < \mathbb{R}_{<0}, \quad \text{and} \quad \{0\} < \mathbb{R}_{>0}.
\]

This simplicial decomposition of \( D^1 \) has 0-simplices,

\[
0, \quad -1, \quad \text{and} \quad 1,
\]

and 1 simplices,

\[
0 \lor -1, \quad \text{and} \quad 0 \lor 1.
\]

The next step is to find the representatives \( z(F, C) \). To do this, set \( y(C_F) \) to be equal to the smallest \( x(C') \) such that \( C' \) is a chamber of \( \mathcal{A} \) contained in \( C_F \). The two chambers of \( \mathcal{A} \) are \( \mathbb{R}_{<0} \) and \( \mathbb{R}_{>0} \), investigate the possible values of \( C_F \). Recall that \( C_F \) is the unique chamber of \( \mathcal{A}_{|F|} = \{H \in \mathcal{A} | H \supseteq X\} \) which contains \( C \), and that \( |F| \) is the span of the facet.

Suppose that \( F = \{0\} \). Then \( |F| \) is just \( \{0\} \) back again and,

\[
\mathcal{A}_{|F|} = \{\{0\}\} = \mathcal{A}.
\]

The chambers of \( \mathcal{A}_{|F|} \) are equal to the chambers of \( \mathcal{A} \). Using the convention for choosing \( y(C_F) \) above, set \( y(\mathbb{R}_{<0}) = -1 \) and \( y(\mathbb{R}_{>0}) = 1 \).

Suppose that \( F = \mathbb{R}_{<0} \) or \( F = \mathbb{R}_{>0} \). Then in both cases \( |F| = \mathbb{R} \), and

\[
\mathcal{A}_{|F|} = \emptyset.
\]

The single chamber of \( \mathcal{A}_{|F|} \) is \( \mathbb{R} \). Since \( C_F \) is the unique chamber of \( \mathcal{A}_{|F|} \) which contains \( C \), it is the case that if \( C = \mathbb{R}_{<0} \) then \( C_F = \mathbb{R} \) and if \( C = \mathbb{R}_{>0} \) then
$C_F = \mathbb{R}$. Using the convention for choosing $y(C_F)$ above, set $y(\mathbb{R}) = -1$.

The values of the $z(F,C)$s (the points which correspond to the 0-simplices of the Salvetti complex) are:

\[
\begin{align*}
z(\mathbb{R}_{<0}, \mathbb{R}_{<0}) &= x(\mathbb{R}_{<0}) + iy(\mathbb{R}), \\
&= -1 + i(-1), \\
&= -1 - i, \\
z(\mathbb{R}_{<0}, \mathbb{R}_{>0}) &= -1 - i, \\
z(\{0\}, \mathbb{R}_{<0}) &= -i, \\
z(\{0\}, \mathbb{R}_{>0}) &= i, \\
z(\mathbb{R}_{>0}, \mathbb{R}_{<0}) &= 1 - i, \\
z(\mathbb{R}_{>0}, \mathbb{R}_{>0}) &= 1 - i.
\end{align*}
\]

From the above ascending chains of facets, the 1-simplices coming from the ascending chain $\{0\} < \mathbb{R}_{<0}$ in $Sal$ correspond to,

\[
-1 - i \lor -i, \text{ and}
\]

\[
-1 - i \lor i,
\]

and the 1-simplices coming from the ascending chain $\{0\} < \mathbb{R}_{>0}$ in $Sal$ correspond to,

\[
1 - i \lor -i, \text{ and}
\]

\[
1 - i \lor i.
\]

Drawing this simplicial complex yields,

\[
\begin{aligned}
&\bullet \\
&\bullet \\
&\bullet \\
&-1 - i \quad -i \quad 1 - i
\end{aligned}
\]

This is clearly homotopic to $S^1$.
6.2 Representing the Salvetti Complex of the Pure Braid Group in Computer Code

As was touched on in the previous section, the classifying space of the pure braid group on \( n \) strands is:

\[
PConf_n \mathbb{C} := \{(v_1, \ldots, v_n) \in \mathbb{C}^n \mid \text{if } i \neq j \text{ then } v_i \neq v_j\}.
\]

This corresponds to the case where the vector space \( V = \mathbb{R} \) and the hyperplane arrangement \( A \) is:

\[
A = \{H_{i,j} \mid 1 \leq i < j \leq n\},
\]

where,

\[
H_{i,j} = \{\vec{v} \in \mathbb{R}^n \mid v_i = v_j\}.
\]

As mentioned earlier, this hyperplane arrangement is not essential, as

\[
\{t(1,1,1,\ldots,1) \mid t \in \mathbb{R}\} \subset \bigcap_{H \in A} H.
\]

In order to construct the Salvetti complex as Paris does in the strictest sense, for this hyperplane arrangement one should project onto a plane perpendicular to the line \((1,1,1,\ldots,1)\). However, the algorithm given in this section will result in an equivalent simplicial complex without doing such a projection, it will also be easier to understand. The drawback of such an approach is that when talking about the ‘dimension’ of a facet here, it will be one greater than if this inessential line had been projected away.

The first task is to choose the \( x(F) \) for each facet \( F \in \mathcal{F}(A) \), these will be a representative for each facet. Construct all the representatives in the following way, beginning with the chambers. Start with the point \( \vec{v} = (1,2,\ldots,n) \in \mathbb{R}^n \), note that \( \vec{v} \) is a point in the chamber \( v_1 < v_2 < \cdots < v_n \). To get a representative for all the other chambers, take the orbit of this element under the symmetric group, \( S_n \). Observe that the orbit of \( \vec{v} \) will be in bijection with the chambers in the hyperplane arrangement \( A \). For example, if one wants the representative of the chamber \( v_3 < v_4 < v_1 < v_5 < v_2 \), then it will be \((3,5,1,2,4)\). This happens to be a specific case of the following theorem.

**Theorem 6.2.1.** Let \( n \in \mathbb{Z}_{\geq 2} \). The codimension \( i \) facets in the hyperplane arrangement associated with the pure braid group on \( n \) strands, \( P\beta_n \), are in
6.2. CODING THE PURE BRAID GROUP

bijection with the set

\[ T_i = \left\{ \vec{x} \in \{1, 2, \ldots, n-i\}^n \mid \text{every } k \in \{1, 2, \ldots, n-i\} \text{ appears in } \vec{x} \text{ at least once} \right\}. \]

Furthermore, exactly one element of \( T_i \) lies in each codimension \( i \) facet.

Proof. Let

\[ T_i = \{ \vec{x} \in \{1, \ldots, n-i\}^n \mid \text{every } k \in \{1, \ldots, n-i\} \text{ appears in } \vec{x} \}. \]

The proof needs to show two things,

1. that every codimension \( i \) facet of \( A \) contains a point from \( T_i \), and
2. that no facet contains two such points.

For the first part, pick a codimension \( i \) facet \( F \) and show that there is a point \( \vec{v} \in F \) such that \( \vec{v} \in T_i \). From the definition of a facet, there exists \( B \subseteq A \) such that \( F \) is a component of

\[ \left( \bigcap_{H \in B} H \right) \setminus \bigcup_{H \in A \setminus B} \left( \bigcap_{H \in B} H \right) \bigcap H'. \]

From this it is clear that,

\[ F \subseteq \bigcap_{H \in B} H, \]

and,

\[ F \cap \left( \bigcup_{H \in A \setminus B} H \right) = \emptyset. \]

That is, \( F \) lies on all the hyperplanes of \( B \), but does not intersect any of the hyperplanes in \( A \setminus B \). Using the hyperplane arrangement, \( A = \{H_{j,k} \mid 1 \leq j < k \leq n\} \) with \( H_{j,k} = \{ \vec{x} \in \mathbb{R}^n \mid x_j = x_k \} \), define for \( \vec{x} \in \mathbb{R}^n \) some sets which will be of use later,

\[ R_\omega(\vec{x}) = \{(j,k) \in \{1, \ldots, n\}^2 \mid x_j = x_k\}, \]
\[ R_\prec(\vec{x}) = \{(j,k) \in \{1, \ldots, n\}^2 \mid x_j < x_k\}, \]
\[ R_\succ(\vec{x}) = \{(j,k) \in \{1, \ldots, n\}^2 \mid x_j > x_k\}, \]
\[ = \{(j,k) \in \{1, \ldots, n\}^2 \mid (k,j) \in R_\prec(\vec{x})\}. \]
That is, $R_\sim(\vec{x})$ is all pairs $(j, k)$ such that $x_j = x_k$, similarly for $R_<(\vec{x})$ and $R_>(\vec{x})$. A subtle point here is that these sets also contain $(j, k)$ with $j > k$, such $H_{j,k}$ are not explicitly in $\mathcal{A}$. However, in this case, using the identification $H_{j,k} = H_{k,j}$ should allay any concerns.

**Claim 6.2.2.** Suppose that $\vec{x}, \vec{y}$ are two points in the same facet, $F$. Then,

1. $R_\sim(\vec{x}) = R_\sim(\vec{y})$,
2. $R_< (\vec{x}) = R_< (\vec{y})$, and
3. $R_>(\vec{x}) = R_>(\vec{y})$.

**Proof of Claim.** Suppose that $(j, k) \in R_\sim(\vec{x})$, so $x_j = x_k$. Therefore, the facet $F$ satisfies $F \subseteq H_{j,k}$, by observations above. Since it was assumed that $\vec{y}$ also lies in $F$, $\vec{y} \in H_{j,k}$, and so $(j, k) \in R_\sim(\vec{y})$. Since this is true for all such pairs, $R_\sim(\vec{x}) = R_\sim(\vec{y})$.

For the second part of the claim, suppose $(j, k) \in R_< (\vec{x})$, so $x_j < x_k$. As $\vec{y}$ is in the same facet as $\vec{x}$, they must both be in the same connected component (by definition of a fact). From the observations above, $F \cap H_{j,k} = \emptyset$. If it were the case that $y_j > y_k$, then the facet $F$ would consist of objects on either side of $H_{j,k}$. However, it is not possible for a set to have objects on either side of a hyperplane, not intersect the hyperplane and still be connected. So $y_j < y_k$ and therefore $(j, k) \in R_<(\vec{y})$. Since this is true for all such pairs, $R_< (\vec{x}) = R_< (\vec{y})$.

The final part of the claim can simply be shown with

$$R_>(\vec{x}) = \{(j, k) \in \{1, \ldots, n\}^2 \mid (k, j) \in R_<(\vec{x})\},$$

$$= \{(j, k) \in \{1, \ldots, n\}^2 \mid (k, j) \in R_< (\vec{y})\},$$

$$= R_>(\vec{y}).$$

$\square$

Pick any $\vec{v} \in F$, since there is a total ordering on $\mathbb{R}$, order the coordinates of $\vec{v}$ as,

$$v_{k_1} \leq v_{k_2} \leq \cdots \leq v_{k_n}, \quad (v_l \in \{1, \ldots, n\}). \quad (6.1)$$

The dimension of $F$ will be the number of $x_i$ which can be moved around at the same time in an independent manner. Specifically, the dimension will be one more than the number of $<$ signs in Equation (6.1). The codimension of $F$ is then the number of $=$ signs in Equation (6.1). For example,

$$v_2 < v_3 = v_1 < v_4 = v_5,$$
will correspond to a facet of dimension $2 + 1 = 3$ and codimension 2 in $\mathbb{R}^5$.

The next step is to construct the $\vec{t} \in T_i$. For the same $\vec{v}$ as above ($\vec{v} = (v_1, \ldots, v_n)$ and $v_{k_1} \leq \cdots \leq v_{k_n}$), use the following process to set the coordinates of $\vec{t}$:

1. For $l = 1$, set the $k_1$th coordinate of $\vec{t}$ to 1.

2. For $l \in \{2, 3, \ldots, n\}$:
   
   (a) If $(k_{l-1}, k_l) \in R_{=}(\vec{v})$ then set the $k_l$th coordinate of $\vec{t}$ to the same value as the $k_{l-1}$th coordinate of $\vec{t}$.
   
   (b) If $(k_{l-1}, k_l) \in R_{<}(\vec{v})$ then set the $k_l$th coordinate of $\vec{t}$ to one greater than the value of the $k_{l-1}$th coordinate of $\vec{t}$.

It is not hard to see that the resulting tuple $\vec{t}$ will be in $T_i$, as it is of length $n$, and

\[
\max(\vec{t}) = n - (\text{the number of } = \text{ signs in Equation [41]}),
\]

\[
= n - (\text{the codimension of } F),
\]

\[
= n - i,
\]

as required. Furthermore, by construction, every $k \in \{1, \ldots, n - i\}$ appears in $\vec{t}$ at least once. Also, note that $R_{=}(\vec{t}) = R_{=}(\vec{v})$ and $R_{<}(\vec{t}) = R_{<}(\vec{v})$ for $\vec{v} \in F$, one should check that this implies that $\vec{t} \in F$. Clearly, if $\vec{v} \in F$, then the line $\alpha \vec{v} + (1 - \alpha)\vec{t}$, $\alpha \in [0, 1]$ is a path from $\vec{v}$ to $\vec{t}$ which has the same $R_{=} \text{ and } R_{<} \text{ at every time point } \alpha$. Therefore, the path does not cross any hyperplanes, and lies entirely in one facet. Thus, $\vec{t} \in F$. The first part is complete, as a $\vec{t} \in T_i \text{ has been constructed which lies in the facet } F$.

Moving on to the second part of the problem, the aim is to show that no two points in $T_i \text{ lie in the same facet.}$ Pick two points $\vec{t}, \vec{t}' \in T_i$ such that there exists $F \in \mathcal{F}(A)$ with $\vec{t}, \vec{t}' \in F$ and $\vec{t} \neq \vec{t}'$. Since $\vec{t}$ and $\vec{t}'$ both lie in the same facet, $R_{=}(\vec{t}) = R_{=}(\vec{t}')$ and $R_{<}(\vec{t}) = R_{<}(\vec{t}')$. Because $\vec{t}, \vec{t}' \in T_i$, it is the case that every $k \in \{1, 2, \ldots, n - i\}$ appears in both $\vec{t}$ and $\vec{t}'$ at least once. By the total ordering on $\mathbb{R}$, write,

\[
1 = t_{i_1} \leq t_{i_2} \leq \cdots \leq t_{i_n} = n - i, \text{ and,}
\]

\[
1 = t'_{i_1} \leq t'_{i_2} \leq \cdots \leq t'_{i_n} = n - i.
\]

Let $j$ be the smallest $j \in \{1, \ldots, n\}$ such that $t_{i_j} \neq t'_{i_j}$. Suppose $t_{i_j} < t'_{i_j}$, then since $t_{i_j}, t'_{i_j} \in \mathbb{Z}$, one of the following possibilities must be true,

1. $t_{i_j} = t_{i_{j-1}}$ and $t'_{i_j} > t'_{i_{j-1}}$ which implies that $R_{=}(\vec{t}) \neq R_{=}(\vec{t}')$, or
2. \(t_{ij} > t_{i,j-1}\) and \(t'_{ij} > t'_{i,j-1} + (t_{i,j-1} - t_{ij})\), which implies that the number \(t'_{i,j-1} + 1 \in \{1, \ldots, n-1\}\) was skipped in \(\vec{v}\), both of which are contradictions to the choice of \(\vec{v}\) and \(\vec{v}'\). Thus, if two in \(T_i\) lie in the same facet then they must be equal.

Having shown shown that every facet contains a point in \(T_i\) for some \(i\), and such a point is unique, the proof is complete.

These elements of \(T_i\) are what will be used for the \(x(F)\) of each facet \(F\). For example, looking at the facet \(x_3 < x_2 = x_4 < x_5 < x_1\), its representative will be \(x(F) = (4, 2, 1, 2, 3)\). Note that this \(F\) is a codimension 1 facet (it lies on exactly one hyperplane: \(x_2 = x_4\)) and indeed, this agrees with the codimension assigned in Theorem 6.2.1.

Having a representative for each facet, the next task is to find the inequalities between the facets of \(A\), as these will eventually define the list of edges, and then higher simplices, in the simplicial complex. Recalling the previous section, the partial order on the facets is defined as \(F_1 < F_2\) if \(F_1 \subseteq F_2\) and \(F_1 \neq F_2\). For \(F_1\) and \(F_2\) facets of the hyperplane arrangement, it is the case that \(F_1 < F_2\) if \(\max(x(F_1)) < \max(x(F_2))\) and \(F_1\) lies on the boundary of \(F_2\). The following proposition defines the order of facets in a more logic driven fashion.

**Proposition 6.2.3.** Let \(F_1\) and \(F_2\) be facets in the hyperplane arrangement for \(P_{\beta_n}\) with \(x(F_1)\) and \(x(F_2)\) as defined above. Then \(F_1 < F_2\) if and only if,

1. \(F_1 \neq F_2\), and

2. if \(i, j \in \{1, \ldots, n\}\) and \(x(F_2)_i \leq x(F_2)_j\) then \(x(F_1)_i \leq x(F_1)_j\).

Note that this also forces the condition: if \(x(F_2)_i = x(F_2)_j\) then \(x(F_1)_i = x(F_1)_j\).

**Proof.** Let \(F_1\) and \(F_2\) be facets of \(A\) such that \(F_1 < F_2\). For the forward direction, the task is to show that for \(i, j \in \{1, \ldots, n\}\), if \(x(F_2)_i \leq x(F_2)_j\) then \(x(F_1)_i \leq x(F_1)_j\). Using the same \(R_\leq(\vec{v})\)s as in the proof of Theorem 6.2.1, namely,

\[
R_\leq(\vec{v}) = \{(j, k) \in \{1, \ldots, n\}^2 \mid v_j = v_k\},
\]

\[
R_\leq(\vec{v}) = \{(j, k) \in \{1, \ldots, n\}^2 \mid v_j < v_k\}.
\]
Set \( \vec{v} = x(F_1) \) and \( \vec{v'} = x(F_2) \). Then the facets satisfy,

\[
F_1 = \begin{cases}
  \vec{z} \in \mathbb{R}^n & \text{if } (i, j) \in R_<(\vec{v}) \text{ then } z_i < z_j, \\
  \quad \text{and} & \text{if } (i, j) \in R_{=}(\vec{v}) \text{ then } z_i = z_j,
\end{cases}
\]

\[
F_2 = \begin{cases}
  \vec{z} \in \mathbb{R}^n & \text{if } (i, j) \in R_<(\vec{v'}) \text{ then } z_i < z_j, \\
  \quad \text{and} & \text{if } (i, j) \in R_{=}(\vec{v'}) \text{ then } z_i = z_j,
\end{cases}
\]

Since \( F_1 < F_2 \), it is the case that \( F_1 \subseteq F_2 \). Looking at \( F_2 \), the closure of \( F_2 \),

\[
F_2 = \begin{cases}
  \vec{z} \in \mathbb{R}^n & \text{if } (i, j) \in R_<(\vec{v'}) \text{ then } z_i \leq z_j, \\
  \quad \text{and} & \text{if } (i, j) \in R_{=}(\vec{v'}) \text{ then } z_i = z_j,
\end{cases}
\]

So in order for \( F_1 \subseteq F_2 \), the following must be satisfied:

1. if \( (i, j) \in R_{=}(\vec{v'}) \) then \( (i, j) \in R_{=}(\vec{v}) \), and
2. if \( (i, j) \in R_<(\vec{v'}) \) then either \( (i, j) \in R_<(\vec{v}) \) or \( (i, j) \in R_{=}(\vec{v}) \).

This is the same as saying that for \( i, j \in \{1, \ldots, n\} \), if \( x(F_2)_i \leq x(F_2)_j \) then \( x(F_1)_i \leq x(F_1)_j \).

Conversely, let \( F_1 \) and \( F_2 \) be two distinct facets of \( A \) such that if \( x(F_2)_i \leq x(F_2)_j \) then \( x(F_1)_i \leq x(F_1)_j \). The aim is to show that \( F_1 < F_2 \). Let \( \vec{v} = x(F_1) \) and let \( \vec{v'} = x(F_2) \). In the proof of Theorem 6.2.1 it was shown that \( R_<(\vec{z}) \) and \( R_{=}(\vec{z}) \) completely determine the facet \( F \) which satisfies \( \vec{z} = x(F) \). As before, the facets satisfy,

\[
F_1 = \begin{cases}
  \vec{z} \in \mathbb{R}^n & \text{if } (i, j) \in R_<(\vec{v}) \text{ then } z_i < z_j, \\
  \quad \text{and} & \text{if } (i, j) \in R_{=}(\vec{v}) \text{ then } z_i = z_j,
\end{cases}
\]

\[
F_2 = \begin{cases}
  \vec{z} \in \mathbb{R}^n & \text{if } (i, j) \in R_<(\vec{v'}) \text{ then } z_i < z_j, \\
  \quad \text{and} & \text{if } (i, j) \in R_{=}(\vec{v'}) \text{ then } z_i = z_j,
\end{cases}
\]

Note that by the assumptions, if \( x(F_2)_i = x(F_2)_j \) then \( x(F_1)_i = x(F_1)_j \) since it is required that both \( x(F_1)_i \leq x(F_1)_j \) and \( x(F_1)_j \leq x(F_1)_i \). So it is the case that \( R_{=}(\vec{v}) \supseteq R_{=}(\vec{v'}) \). Secondly, the remainder of the argument says that if \( x(F_2)_i < x(F_2)_j \) then either \( x(F_1)_i < x(F_1)_j \) or \( x(F_1)_i = x(F_1)_j \). This implies that if \( (i, j) \in R_<(\vec{v}) \) then either \( (i, j) \in R_<(\vec{v'}) \) or \( (i, j) \in R_{=}(\vec{v}) \). Again, looking at \( F_2 \),

\[
F_2 = \begin{cases}
  \vec{z} \in \mathbb{R}^n & \text{if } (i, j) \in R_<(\vec{v'}) \text{ then } z_i \leq z_j, \\
  \quad \text{and} & \text{if } (i, j) \in R_{=}(\vec{v'}) \text{ then } z_i = z_j
\end{cases}
\]

It is then clear that \( F_1 \subseteq F_2 \), and \( F_1 < F_2 \) since \( F_1 \) and \( F_2 \) were assumed to be distinct. \qed
CHAPTER 6. THE SALVETTI COMPLEX

To code this information on a computer, start with the chambers (codimension 0 facets) being the elements of $S_n$ (as all permutations of $(1, 2, \ldots, n)$), then apply an algorithm which will generate the higher codimension facets from them. The following algorithm will construct all the codimension $i + 1$ facets $F'$ from a given codimension $i$ facet $F$ such that $F' < F$:

**Algorithm 6.2.4.**

old_facet <- input
new_facets <- empty list

for $i = 2$ to maximum coordinate value in old_facet:
    new_facet <- old_facet
    for each coordinate in new_facet:
        if value of the coordinate is greater than or equal to $i$,
            then reduce its value by 1.
    add new_facet to list new_facets

return new_facets

This algorithm effectively increases the ‘number of equalities’ in old_facet. For example, iterating the above algorithm will give the following relations,

$$(1, 2, 3, 4, 5) > (1, 2, 2, 3, 4) > (1, 1, 1, 2, 3) > (1, 1, 1, 2, 2) > (1, 1, 1, 1, 1).$$

At each step, the $x(F)$ being looked at is on more hyperplanes than the previous step, and thus corresponds to a higher codimension facet. Think of the $x(F)$ as representing the facet which contains every point in $\mathbb{R}^n$ with the same ‘inequality profile’ as the $x(F)$. For example, the point $(1, 2, 2, 3, 4)$, above, represents the set,

$$\{(x_1, x_2, \ldots, x_5) \in \mathbb{R}^n \mid x_1 < x_2 = x_3 < x_4 < x_5\}.$$ 

All the inequalities from codimension $i$ to codimension $i + 1$ have been constructed for each $i$. This does not produce the relations for general codimensions (that is, for example, whether $F_1 < F_2$ where $F_1$ is codimension 4 and $F_2$ is codimension 2). To generate these remaining relations, start with the current list of relations, and add to the list until the following is satisfied for all facets $F_1, F_2$ and $F_3$:

if $F_1 < F_2$ and $F_2 < F_3$ then $F_1 < F_3$.

After doing this, the path to the Salvetti complex requires the complexifica-
tion of the points. In the previous section, this is done by choosing \( y(C_F) \) for each possible \( C_F \), where \( C \) is a chamber and \( F \) is a facet. Recall that \( C_F \) is defined as follows:

- Let \( A_{|F|} \subseteq A \) be the hyperplane arrangement

\[
A_{|F|} = \{ H \in A \mid |F| \leq H \},
\]

where \( |F| \) is the span of the facet \( F \). In the current setting, \( A_{|F|} \) is precisely the set of all hyperplanes which contain \( x(F) \).

- Then \( C_F \) is the chamber in the hyperplane arrangement \( A_{|F|} \) which contains \( C \).

From before, a set in bijection with the chambers of \( A \) has already been constructed, namely all permutations of \((1, 2, \ldots, n)\). The task is to find a unique representative for each of the \( C_F \). In order to do this, start with the facet \( F \) and the chamber \( C \). Start with the same representative for \( C \) which was used for the facets above (that is \( x(C) \)). The \( x(F) \) lies in some number of hyperplanes. By the construction of \( C_F \), the hyperplanes which contain \( x(F) \) persist on to form the boundaries of \( C_F \), the other hyperplanes are ignored. This means that the representative for \( C \) can be ‘moved around’, so long as it doesn’t cross any of the hyperplanes containing \( x(F) \), doing this results in a point still in \( C_F \). However, the representative \( y(C_F) \) needs to be chosen so that it can be reconstructed, so that whenever \( C_F = C'_F \) for some chambers \( C, C' \) and facet \( F \) it is the case that \( y(C_F) = y(C'_F) \). To do this, find the ‘smallest’ chamber representative \( y(C_F) \) for \( C_F \) such that if \( x(F)_i = x(F)_j \) and \( x(C)_i < x(C)_j \) then \( y(C_F)_i < y(C'_F)_j \). Where ‘smallest’ means that if \( Y \) also satisfies the above conditions for \( y(C_F) \), and \( i \) is the first coordinate at which they differ, then \( y(C_F)_i < Y_i \).

The algorithm used to find the \( y(C_F) \) is as follows, with \( facet = x(F) \) and \( chamber = x(C) \):

**Algorithm 6.2.5.**

```plaintext
F <- input facet
C <- input chamber
n <- length of F = length of C
CF <- (0,0,...,0) (length n)

for i = 1 to n:
    lst <- the list of coordinates j < i such that:
        F[i] = F[j] and C[i] < C[j]
    if lst is non-empty:
        CF[i] = \( \min \{ y \in \mathbb{R} \mid y \in \text{lst} \} \)
```

6.2. CODING THE PURE BRAID GROUP
min_val <- smallest value of a coordinate of CF which is listed in lst
increase all values of coordinates in CF which have value greater than or equal to min_val by 1
set CF[i] = min_val
else:
set CF[i] = i
return CF (This is the y(C_F))

The accuracy of this algorithm should hopefully become clear while working through Example 6.2.7.

Now that the representatives for all the facets and chambers have been found, it is time to construct the simplicial complex. Start by defining the vertices of the simplicial complex $Sal$ to be represented by

$$z(F, C) := (x(F), y(C_F)),$$

where $F$ is a facet and $C$ is a chamber. The partial ordering between the facets in $A$ has already been found, this gives the corresponding orderings between the Salvetti complex vertices. This is achieved by saying, if $F_1 < F_2$ and $C$ is a chamber, then $z(F_1, C) < z(F_2, C)$ on vertices. To get the $k$-simplices of the Salvetti complex, one simply takes all possible $(k+1)$-chains of the form $z(F_0, C) < z(F_1, C) < z(F_2, C) < \cdots < z(F_k, C)$. The corresponding $k$-simplex is $z(F_0, C) \vee z(F_1, C) \vee \cdots \vee z(F_k, C)$, with vertices $z(F_i, C)$.

The boundary of a simplex is defined in the standard way,

$$d(\langle v_0, v_1, \ldots, v_k \rangle) = \sum_{i=0}^{k} (-1)^i \langle v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k \rangle.$$

Proceed by putting the boundary maps at each dimension into matrix form and calculate their ranks and nullities in order to find the Betti numbers for the simplicial complex.

Indeed, for example, doing the above construction with $n = 4$, the following Betti numbers are obtained for the Salvetti complex:

$$b_i(Sal) = \begin{cases} 1, & i = 0, \\ 6, & i = 1, \\ 11, & i = 2, \\ 6, & i = 3, \\ 0, & i \geq 4, \end{cases}$$
Observe that by the following theorem of Arnol’d, [Arn69], these are indeed the Betti numbers of the pure braid group on 4 strands.

**Theorem 6.2.6 ([Arn69]).** The cohomology of the pure braid group on \( n \) strands is the same as the cohomology of the direct product of the following spaces,

- a circle;
- a bouquet of two circles;
- ...
- a bouquet of \( n - 1 \) circles.

By the Künneth formula, the cohomology of the pure braid group on \( n \) strands is,

\[
H^*(P\beta_n) = H^*(S^1) \otimes H^*(S^1 \vee S^1) \otimes \cdots \otimes H^*\left(\bigvee_{i=1}^{n-1} S^1\right).
\]

Apply this theorem in the case of \( n = 4 \). The theorem says that

\[
H^*(P\beta_4) = H^*(S^1) \otimes H^*(S^1 \vee S^1) \otimes H^*(S^1 \vee S^1 \vee S^1).
\]

Write the generators for the cohomologies on the right-hand-side as

\[
a \in H^*(S^1), \quad b, c \in H^*(S^1 \vee S^1), \quad d, e, f \in H^*(S^1 \vee S^1 \vee S^1).
\]

Writing \(*\) for the trivial element of the cohomology in each case, the single generator for \( H^0(P\beta_4) \) is,

\[
* \otimes * \otimes *.
\]

the 6 generators for \( H^1(P\beta_4) \) are,

\[
a \otimes * \otimes *, \quad * \otimes b \otimes *, \quad * \otimes c \otimes *, \quad * \otimes * \otimes d, \quad * \otimes * \otimes e, \quad * \otimes * \otimes f.
\]
the 11 generators for $H^2(P_{4})$ are,

\[
\begin{align*}
& a \circ b \circ *, \ a \circ c \circ *, \\
& a \circ * \circ d, \ a \circ * \circ e, \ a \circ * \circ f, \\
& * \circ b \circ d, \ * \circ b \circ e, \ * \circ b \circ f, \\
& * \circ c \circ d, \ * \circ c \circ e, \ * \circ c \circ f,
\end{align*}
\]

and the 6 generators for $H^3(P_{4})$ are,

\[
\begin{align*}
& a \circ b \circ d, \ a \circ b \circ e, \ a \circ b \circ f, \\
& a \circ c \circ d, \ a \circ c \circ e, \ a \circ c \circ f.
\end{align*}
\]

Having found the generators for the cohomology of the pure braid group on 4 strands, the Betti numbers are simply,

\[
\begin{align*}
b_i(P_{4}) = \begin{cases} 
1, & i = 0, \\
6, & i = 1, \\
11, & i = 2, \\
6, & i = 3, \\
0, & i \geq 4,
\end{cases}
\end{align*}
\]

which agrees with the Betti numbers obtained from the Salvetti complex.

**Example 6.2.7** ($n = 3$). An example of completing this process when $n = 3$. The hyperplane arrangement is made up of three hyperplanes in $\mathbb{R}^3$, namely $\{x_1 = x_2\}, \{x_2 = x_3\}, \{x_3 = x_1\}$. As noted before, the line $\{(t, t, t) | t \in \mathbb{R}\}$ lies in all hyperplanes, projecting onto a plane perpendicular to this line makes the arrangement essential. The pictures throughout this example will be drawn as if the reader is looking down the $(1, 1, 1)$ line. The hyperplane arrangement is shown in Figure 6.1.

Choosing the representatives for each facet using the technique described above results in the labelling in Figure 6.2. Looking at the chamber represented by $(1, 2, 3)$, the partial order on the facets produces the picture in Figure 6.3 (where the arrows point from the larger facet to the smaller facet). This gives the following relations on the facets,

\[
\begin{align*}
(1, 2, 3) &> (1, 1, 2) > (1, 1, 1), \\
(1, 2, 3) &> (1, 2, 2) > (1, 1, 1), \\
(1, 2, 3) &> (1, 1, 1).
\end{align*}
\]

Note that the same relations are obtained by recursively applying Algorithm
6.2. CODING THE PURE BRAID GROUP

Figure 6.1: The hyperplane arrangement corresponding to the pure braid group on three strands.

Figure 6.2: The facet representatives corresponding to the pure braid group on three strands.
**CHAPTER 6. THE SALVETTI COMPLEX**

Figure 6.3: The ordering of the facets in the Salvetti complex for the pure braid group on three strands.

Extending these relations so that \( > \) is a partial order on the set of facets gives the missing relation \((1, 2, 3) > (1, 1, 1)\).

The next part of the process is to choose the \( y(C_F) \) representatives as described above. One should think of the process as picking a facet \( F \), then defining all the representatives \( y(C_F) \) for the \( C_F \) obtained from that facet. If \( F \) lies on a hyperplane, that hyperplane will ‘persist’ through to the part where the \( y(C_F) \) are chosen. The hyperplanes which do not contain \( F \) do not survive through to the end and causes the adjacent chambers to merge. Some examples follow which should make the process clearer.

If \( F = (1, 1, 1) \), then all of the hyperplanes persist through to the \( C_F \) stage. This means that the \( y(C_F) \) will equal \( x(C) \) (See Figure 6.4). If \( F = (2, 1, 1) \), only the \( \{x_2 = x_3\} \) hyperplane persists through to the \( C_F \) stage, the others play no role. In this case there are only two possible \( y(C_F) \). If the chamber \( C \) is on the \( x_2 > x_3 \) side of the hyperplane, then \( y(C_F) = (1, 3, 2) \). If the chamber \( C \) is on the \( x_2 < x_3 \) side of the hyperplane, then \( y(C_F) = (1, 2, 3) \). Note that these choices of \( y(C_F) \) satisfy the ‘smallest representative’ property defined above. A picture of this case is shown in Figure 6.5. If the facet \( F \) happens to be any chamber, then no hyperplanes persist through to the \( C_F \) stage. There is only
6.2. CODING THE PURE BRAID GROUP

Figure 6.4: The representatives of the chambers $C_F$ when $F$ is the centre, $F = (1,1,1)$.

Figure 6.5: The representatives of the chambers $C_F$ when $F = (2,1,1)$ is a codimension 1 facet.
one $y(C_F)$, namely $(1, 2, 3)$ (the ‘smallest’ representative of any chamber), as seen in Figure 6.6.

Define the points,

$$z(F, C) := (x(F), y(C_F)),$$

and define $z(F, C) < z(F', C)$ whenever $F < F'$ for $F, F'$ facets and $C$ a chamber. For example, if $C = (3, 2, 1)$ and there is a chain of facets $(3, 2, 1) > (2, 1, 1) > (1, 1, 1)$, there will be a chain of $z(F, C)$s:

$$((3, 2, 1), (1, 2, 3)) > ((2, 1, 1), (1, 3, 2)) > ((1, 1, 1), (3, 2, 1)).$$

This corresponds to a 2-simplex with vertices represented by the three $z(F, C)$.

Unfortunately the remainder of the process becomes too hard to be done by hand and is better left to a computer. The Salvetti complex in this case has 24 vertices, 96 edges and 72 faces. The resulting Betti numbers for this complex is as follows,

$$b_i(Sal) = \begin{cases} 1, & i = 0, \\ 3, & i = 1, \\ 2, & i = 2, \\ 0, & i \geq 3. \end{cases}$$

Note that this is the same result as obtained when applying Theorem 6.2.6.
above to calculate the Betti numbers for the pure braid group on 3 strands.

The computer code to calculate these Betti numbers for the pure braid group on $n$ strands can be found in Appendix A.
Bibliography


Appendix A

Python Code to Calculate the Betti Numbers of the Pure Braid Group

This appendix contains the code used to calculate the Betti numbers for the pure braid group. The computer code is written in Python 2 and utilises the *numpy* package \[^{[JOP]}\], version 1.7.0b2.

For \( n = 3 \), the output is:

For the pure braid group on 3 strands:
The 0 Betti number is: 1
The 1 Betti number is: 3
The 2 Betti number is: 2

and for \( n = 4 \), the output is:

For the pure braid group on 4 strands:
The 0 Betti number is: 1
The 1 Betti number is: 6
The 2 Betti number is: 11
The 3 Betti number is: 6

It should be noted that higher calculations are infeasible. The calculation for the pure braid group on 3 strands takes less than one tenth of a second. However, the calculation for the pure braid group on 4 strands takes a bit over three minutes.\(^1\) A successful run for 5 strands has not been completed. Because

\(^1\)Times taken refer to how long the program took to run on an *Intel(R) Core(TM) i7-2600 CPU @ 3.40GHz* processor.
of this, the program should be thought of as more of a proof-of-concept, rather than being used for any serious calculations.

The slowest part of the program is the calculation of the rank of the boundary matrices. Attempts to speed this up proved fruitless. Any improvement in speed was offset by a reduction in the accuracy of the calculations, introducing rounding errors and giving incorrect matrix ranks. This area remains a potential place to make calculation optimisations.

A.1 The Code

code/pureBraid.py

```python
#!/usr/bin/env python2

import itertools
import numpy

def get_chambers(n):
    """ Gets a list of representatives for the chambers of a hyperplane arrangement with n hyperplanes. (There should be (n)! chambers where n is the # of strands in the braid.) """
    # The representatives should be the orbit of [0, 1, ..., n - 1] under the symmetric group.
    return symmetric_orbit(range(n))

def symmetric_orbit(element):
    """ Returns a list of the orbit of the input under the action of the symmetric group. ie, all orderings of the elements. """
    # 'dirty' will contain elements that haven't been checked yet.
    dirty = []
    # 'elements' will be a list of all elements in the orbit which have been found so far.
    elements = []
    # First, add the passed in element to both lists.
    dirty.append(element)
    elements.append(element)
    # While there are still unchecked items, pick off the next one.
```
while len(dirty) > 0:
    to_check = dirty[0]
    dirty.remove(to_check)
    # Apply all the simple transpositions to to_check.
    for i in range(len(element) - 1):
        new = to_check[:i] \n        + [to_check[i+1], to_check[i]] \n        + to_check[i+2:]
        # If this 'new' element hasn't been found before, add it
        # to both lists. Otherwise, ignore it.
        if new not in elements:
            elements.append(new)
            dirty.append(new)

return elements

def get_edges(n):
    ''' This will get the edges between facets of the arrangement.
    That is, the < relation. To do this, get all the
    codimension 0 facets, then increase the number of
    equalities in the representative to get a higher co-dimension
    facet connected by an edge. '''
    # The first step is to generate the co-dimension 0 facets.
    chambers = symmetric_orbit(range(n))
    edges = {}

    for facet in chambers:
        # Generate all the edges from the chamber. The way that the
        # function get_edges_from_facet is defined means that just
        # doing this on the chambers is enough to find all edges in
        # the simplicial complex.
        edges.update(get_edges_from_facet(facet, edges)[1])

    return edges

def get_edges_from_facet(f, e):
    ''' The inputs are:
    f - a facet; and
    e - a dictionary of edges which are already known / found.
This function aims to find all facets that $f$ has $<$ signs to.

The output is $(a, b)$ where:

- $a$ is a list of all lower dimension facets $< f$; and
- $b$ is a dictionary describing all the edges found as a consequence of this function.

In practice, this function will only be run on the codimension 0 facets, as $e$ will contain all the edges needed.

```
# facetsl is a list of facets which have an edge to the facet f.
facetsl = []
# facetsd is a dictionary which consists of
# keys - a facet connected to f by an edge; and
# values - all facets connected to the key by an edge.
facetsd = {}

# The base case, return no edges if at the codimension n facet.
if max(f) == 0: return (facetsl, facetsd)
# Otherwise, go through each of i in (1, 2, ..., facet.max()).
# Then reduce every value $>= i$ in f by 1 and add it to the
# list of facets connected to f by an edge.
for i in range(1, max(f)+1):
    facet = list(f)
    for j in range(len(facet)):
        if facet[j] >= i:
            facet[j] = facet[j] - 1
        # Add the new facet to the list of facets.
tfacet = tuple(facet)
facetsl.append(tfacet)
# Get all the lower dimension facets from tfacet by either
# calculating them or by using already calculated ones.
if tfacet in e.keys():
    1 = e[tfacet]
else:
    # The recursive step, call this function again on the
    # lower facet.
    (1, d) = get_edges_from_facet(facet, e)
facetsd.update(d)
e.update(d)
# Add the lower facets to the list of facets connected to
# f by an edge.
```
A.1. THE CODE

```python
facetsl = facetsl + 1
# The list(set) function removes duplicates.
facetsd[tuple(f)] = list(set(facetsl))

return (facetsl, facetsd)

def take_power_sets(lists):
    ''' lists is a list of lists, take the power set of each list
    and return a list of all of them. '''
powerset = []

    for l in lists:
        for i in range(1, len(l) + 1):
            for x in itertools.combinations(l, i):
                # Append all possible subsets of l to the list.
                powerset.append(x)

    # Return the list after removing duplicates.
    return list(set(powerset))

def get_simplices_chambers(n):
    ''' n is the number of strands in the pure braid group.
    The output is the simplices of the corresponding Salvetti complex. The returned object is a list of lists, each
    of these lists corresponds to an m simplex (a list of
    length m + 1, elements of the form (x(F), y(C_F)). '''
    # Start with the list of chambers, and the dictionary of all
    # edges between facets.
    chambers = get_chambers(n)
    edges = get_edges(n)
    # Get the 'first' simplices. These aren't really simplices,
    # but are more an chain of < relations on the facets of
    # the hyperplane arrangement for the pure braid group. But
    # certainly not all of them.
    simplices = get_first_simplices(n)
    # This list will hold the simplices with vertices of the
    # form (x(F), y(C_F)).
    simplices_chamber = []
```
for chamber in chambers:
    for simplex in simplices:
        # For each ‘simplex’, go through and calculate
        # (x(F), y(C_F)) for each facet at a vertex.
        # This gives an (n-1)-simplex of the Salvetti complex.
        simplex_chamber = []
        for facet in simplex:
            simplex_chamber.append(((facet, get_C_F_from_facet_and_chamber(facet, chamber)))
            simplices_chamber.append(simplex_chamber)

    # At this point, simplices_chamber contains all of the (n-1)
    # dimensional simplices of the Salvetti complex. Taking the
    # power sets of these will give the lower dimensional
    # simplices as well.
    simplices_chamber = take_power_sets(simplices_chamber)

return simplices_chamber

def get_first_simplices(n, prefix=False, edges=False):
    """This is a recursive function which will give out a list
    of some ‘pre–simplices’ from the hyperplane arrangement
    corresponding to the pure braid group on n strands.
    The return value is a list of tuples, containing a chain
    of facet representatives which are related by > (relation
    on the facets in the definition of hyperplane arrangement).
    This will only find the (n – 1)–‘simplices’, taking the
    power sets in another function later will find the missing
    simplices.""
    # Start by getting all the edges, this returns a dictionary.
    # There is one key for each vertex, the values are a list of all
    # vertices which the key vertex has an edge to.
    if not edges: edges = get_edges(n)

    # When referring to a vertex as being either higher up, or lower,
    # this is in terms of the highest number in the tuple describing
    # the vertex.

    # Start with no simplices.
    simplices = []
# prefix holds a list of higher up vertices which were already
# explored on the way down. This effectively does a search
# down the chain.
if prefix:
    # If this is not the first vertex visited, first check if
    # the end has been reached 'the base case'. (If that’s the
    # case, prefix[-1] should be (0,0,0,...,0).)
    # If the end has been reached, then prefix is an
    # (n - 1)-simplex. Start heading back up the chain.
    if prefix[-1] not in edges.keys(): return [prefix]
    # If not at the centre yet, explore all the facets of one
    # less dimension which are connected to the current facet
    # by an edge.
    e = edges[prefix[-1]]
    # Filter out everything which is not of one less dimension.
    to_check = filter(lambda x: max(x) == max(prefix[-1]) - 1, e)
else:
    # If this is the first vertex visited, get the list of
    # facets to explore from the list chambers. (That is, keys
    # with maximum element = n - 1).
    to_check = filter(lambda x: max(x) == n - 1, edges.keys())
for x in to_check:
    # For each facet that needs to be checked, create a new
    # prefix for the next time this function runs. The new
    # prefix is the old prefix with the current facet added
    # at the end. If prefix is False (i.e. this is the first
    # time this function is being run), set the entire prefix
    # to be simply the current facet.
    new_prefix = prefix + [x] if prefix else [x]
    # To get the simplices which start with new_prefix, call
    # this function again, with the new prefix.
    simplices += get_first_simplices(n, new_prefix, edges)
return simplices

def get_boundary_of_simplex(simplex):
    """ Calculates the boundary of the simplex. This is simply,
For each vertex, forget vertex 1 - vertex 2 + vertex 3 -...

''

boundary = []
multiplier = 1

for x in simplex:
    # Go through each vertex of the simplex, create a copy of
    # the set of original vertices, then remove the corner
    # which was picked.
    s = list(simplex)
    s.remove(x)
    # Add this dimension n - 1 simplex to the list of boundary
    # simplices, with the appropriate multiplier.
    boundary.append((multiplier, tuple(s)))
    # Flip the multiplier sign each time so that the
    # +1, -1, +1, -1 pattern happens.
    multiplier *= -1

return boundary

def get_C_F_from_facet_and_chamber(facet, chamber):
    """ Takes in a facet and a chamber of the hyperplane
    arrangement and returns y(C,F). """
    n = len(facet)

    # Construct the smallest possible representative that’s in the
    # chamber C,F.
    # Start the representative at [0,0,0,0,...].
    rep = [0]*n

    for i in range(n):
        # lst is a list of all previous coordinates which the ith
        # coordinate should be less than.
        lst = []
        for j in range(i):
            # Check all the coordinates before i, it’s not
            # possible to swap coordinates if facet[i] and
            # facet[j] are the same. If they are the same, the
            # inequality between chamber[i] and chamber[j] should
            # not change.
A.1. THE CODE

if (facet[i] == facet[j] and
    chamber[i] < chamber[j]):
    lst.append(j)

# Now apply these inequalities to the representative.
# If there are relations that are required to be present in
# y(C_F).
if lst:
    # The minimum value of all the coordinates which are
    # being increased will be the entry for rep[j].
    min_val = rep[lst[0]]
    for j in lst:
        # Run through all things coordinate i should be less
        # than and keep the smallest one.
        if min_val > rep[j]: min_val = rep[j]
    for j in range(i):
        # Increase all the coordinates which have values
        # greater than or equal to the minimum value in order
        # to make room for rep[i] to equal min_val.
        if rep[j] >= min_val: rep[j] += 1
    # rep[i] is set to the minimum value that was increased.
    rep[i] = min_val
else:
    # If there are no relations to worry about, just set
    # rep[i] to i.
    rep[i] = i

# Return as a tuple.
return tuple(rep)

def calc_boundary_matrices_rank_nullities_of_pb_n(n):
    '''This function takes the number of strands in the pure braid
    group, puts the boundary maps on simplices into matrices,
    then returns the rank and nullities of these matrices.
    Output is [(rank d1, null d1), (rank d2, null d2), ...].'''
simplices = get_simplices_chambers(n)

    # For each simplex in simplices, it's dimension will be
    # len(simplex) – 1. The aim is to enumerate each dimension's
    # simplices. This object will be a dictionary to hold the
APPENDIX A. THE BETTI NUMBERS OF $P^2_N$ IN PYTHON

```python
# representatives of each simplex. This is graded by simplex
# dimension.
# This loop is required because reps = [{}]*n gives n copies of
# the SAME dictionary.
reps = []
for i in range(n):
    reps.append({})
# This list will hold the next number which will be assigned to
# each dimension simplex.
next_id = [0]*n

for simplex in simplices:
    # Enumerate all the simplices so that no two simplices of
    # the same dimension have the same id.
    simplex_dim = len(simplex) - 1
    reps[simplex_dim][simplex] = next_id[simplex_dim]
    next_id[simplex_dim] += 1

# Now construct the matrices for the boundary maps.
# Note that the boundary maps should technically go in as
# the columns. This is rectified later in this function.
M_list = [0]*n
for i in range(n):
    # Find how many rows the matrix should have.
    row_len = 1 if i == 0 else len(reps[i-1].keys())
    M = [0]*len(reps[i].keys())
    for simplex in reps[i].keys():
        # Want to encode the boundary of the simplex for each
        # dimension i-1 simplex.
        row = [0]*row_len
        if i > 0:
            # bound elt[1] contains the simplex with one less
            # vertex. bound elt[0] is either +1 or -1.
            for bound_elt in get_boundary_of_simplex(simplex):
                row[reps[i-1][bound_elt[1]]] += bound_elt[0]
        # Add the new row into the matrix.
        M[reps[i][simplex]] = row
    M_list[i] = M

# With the boundary matrices, calculate the rank and nullity of
```
A.1. THE CODE

```python
# each.
# This list will hold the rank and nullity of d_i.
d_i_rank_nullity = []
for i in range(n):
    # Use numpy to calculate the rank.
    rank = numpy.linalg.matrix_rank(numpy.matrix(M_list[i]))
    # nullity is #columns - rank.
    # Note that len(M_list[i]) is going to give the number of
    # rows in the matrix. However, the matrices here are the
    # transpose of the actual boundary matrices as noted above.
    nullity = len(M_list[i]) - rank
d_i_rank_nullity.append((rank, nullity))
return d_i_rank_nullity

def calc_betti_of_pb_n(n):
    ''' This function calculates the Betti numbers of the pure braid
    group on n strands.'''
    # The Betti numbers are simply the nullity of the dimension
    # k matrix minus the rank of the k+1 matrix.
    rank_null = calc_boundary_matrices_rank_nullities_of_pb_n(n)
betti_no = []
for i in range(len(rank_null)):
    try:
        betti_no.append(rank_null[i][1] - rank_null[i + 1][0])
    except IndexError:
        betti_no.append(rank_null[i][1])
return betti_no

# This code here prints out the Betti numbers for the pure braid
# group on 4 strands.
n = 4
betti = calc_betti_of_pb_n(n)
print "For the pure braid group on %d strands:" % n
for i in range(len(betti)):
    print "The %d Betti number is: %d" % (i, betti[i])
```
Minerva Access is the Institutional Repository of The University of Melbourne

Author/s:
Bailes, Jeffrey

Title:
Orbispaces, configurations and quasi-fibrations

Date:
2015

Persistent Link:
http://hdl.handle.net/11343/57345