RESEARCH ARTICLE

Synthesis of control Lyapunov functions and stabilizing feedback strategies using exit-time optimal control

Part I: Theory

Ivan Yegorov*1 | Peter M. Dower2 | Lars Grüne3

1Department of Mathematics, North Dakota State University, North Dakota, USA
2Department of Electrical and Electronic Engineering, University of Melbourne, Victoria, Australia
3Chair of Applied Mathematics, University of Bayreuth, Bavaria, Germany

Abstract

This work studies the problem of constructing control Lyapunov functions (CLFs) and feedback stabilization strategies for deterministic nonlinear control systems described by ordinary differential equations. Many numerical methods for solving the Hamilton–Jacobi–Bellman partial differential equations specifying CLFs typically require dense state space discretizations and consequently suffer from the curse of dimensionality. A relevant direction of attenuating the curse of dimensionality concerns reducing the computation of the values of CLFs and associated feedbacks at any selected states to finite-dimensional nonlinear programming problems. We propose to use exit-time optimal control for that purpose. This paper is the first part of a two-part work. First, we state an exit-time optimal control problem with respect to a sublevel set of an appropriate local CLF and establish that, under a number of reasonable conditions, the concatenation of the corresponding value function and the local CLF is a global CLF in the whole domain of asymptotic null-controllability. We also investigate the formulated optimal control problem. A modification of these constructions for the case when one does not find a suitable local CLF is provided as well. Our developments serve as a theoretical basis for a curse-of-dimensionality-free approach to feedback stabilization, that is presented in the second part of this work together with supporting numerical simulation results.

KEYWORDS:
control Lyapunov functions, feedback stabilization, exit-time optimal control, Hamilton–Jacobi–Bellman equations, curse of dimensionality, Pontryagin’s principle, characteristic Cauchy problems.

1 | INTRODUCTION

In control theory and engineering, feedback stabilization methods for nonlinear dynamical systems are of both theoretical and practical importance, and control Lyapunov functions (CLFs) constitute a fundamental tool there [2–10]. As was established in [5] for a relatively wide subclass of deterministic control systems described by ordinary differential equations (ODEs) without state constraints, the value functions of appropriate infinite-horizon optimal control problems are CLFs and also the unique viscosity solutions of boundary value problems for the corresponding Hamilton–Jacobi–Bellman (HJB) partial differential equations (PDEs) of first order. This is in fact an extension of the classical Zubov method for finding Lyapunov functions [11].
to problems of weak asymptotic null-controllability. Moreover, the framework of [5] can be extended to some state-constrained problems (see [12] Example 5.2).

Exact solutions of boundary value, initial value, and mixed problems for HJB equations are known only in very special cases. Many broadly used numerical approaches to solving these problems, including semi-Lagrangian schemes [13–17], finite-difference schemes [17–24], finite element methods [25], and level set methods [26–31], typically rely on dense state space discretizations. With the increase of the state space dimension, the computational cost of such grid based techniques grows exponentially. Their practical implementation is in general extremely difficult (even on supercomputers) if the state space dimension is greater than 3, which leads to what R. Bellman called the curse of dimensionality [32, 33]. Possible ways to attenuate the curse of dimensionality for various classes of HJB equations and also more general Hamilton–Jacobi (HJ) equations, such as Hamilton–Jacobi–Isaacs (HII) equations for zero-sum two-player differential games, have therefore become an important research area. A number of related approaches have been developed for particular classes of problems (see, e.g., the corresponding overview in [34] Introduction and Section 4). It has to be emphasized that, even when the curse of dimensionality is mitigated, the so-called curse of complexity may still cause significant issues in numerical implementation [34–50].

A relevant direction of attenuating the curse of dimensionality for certain classes of first-order HJ equations is reducing the evaluation of their solutions at any selected states to finite-dimensional optimization (nonlinear programming) problems [34, 36–39]. In contrast with the aforementioned grid based techniques, this leads to the following advantages:

- the solutions can be evaluated independently at different states, which allows for mitigating the curse of dimensionality;
- since different states are separately treated, one can choose arbitrary bounded regions and grids for computations and arrange parallelization;
- when obtaining the value functions, i.e., the solutions of HJB or HII equations, at selected states by solving the related finite-dimensional optimization problems, one can usually retrieve the corresponding costates and control actions as well, without requiring possibly unstable approximations of the partial derivatives of the value functions.

However, the curse of complexity still takes place if the considered nonlinear programming problems are essentially multi-extremal or if one wants to construct global solution approximations in high-dimensional regions.

The finite-dimensional optimization problems describing the values at arbitrary isolated states of the solutions of first-order HJB equations in optimal control problems may build on the (generalized) method of characteristics for such PDEs [34, 36, 38] (related also to Pontryagin's principle [40–42]), or on so-called direct approximation techniques [43–52]. The latter involve direct transcriptions (approximations) of infinite-dimensional optimal open-loop control problems to finite-dimensional nonlinear programming problems via discretizations in time applied to state and control variables, as well as to dynamical state equations. In this context, the frameworks based on Pontryagin’s principle and the method of characteristics are called indirect. In comparison, the direct numerical approaches are in principle less precise and less justified from a theoretical perspective, but often more robust with respect to initialization and more straightforward to use.

For designing a curse-of-dimensionality-free approach to feedback stabilization in one of the ways discussed above, it is crucial first to bridge the gap between the infinite-horizon Zubov type setting of [5] and numerical optimization frameworks handling only finite terminal (exit) times. To that end, one can impose an appropriate terminal condition leading to an exit-time optimal control problem. Such a formulation is in particular involved in the work [53] developing model predictive control (MPC) schemes for stabilization, while some other MPC studies, such as [54–56], use terminal conditions with fixed horizon length. In general, the works [53, 55, 56] adopt local asymptotic controllability conditions and establish the existence of sufficiently small sampling times and sufficiently large prediction horizons such that systems driven by the corresponding MPC algorithms become asymptotically stable for given initial states.

In comparison, our work establishes global characterizations of CLFs via exit-time optimal control, serving as a theoretical basis for curse-of-dimensionality-free approaches to feedback stabilization. It extends the results of our conference papers [57, 58] and provides detailed proofs, discussions, and practical developments.

This paper is the first part of the two-part work and focuses on theoretical developments. The second part [1] contains the description of a related algorithmic curse-of-dimensionality-free approach to feedback stabilization together with supporting numerical simulation results.

The paper is organized as follows. In Section 2, we state an exit-time optimal control problem with respect to a sublevel set of an appropriate local CLF similarly to [53]. It is then shown that, under a number of reasonable conditions, the concatenation of the corresponding value function and the local CLF is a global CLF in the whole domain of asymptotic null-controllability. We also investigate the formulated problem and derive a characteristics based representation of the value function. Section 3
presents a modification of these constructions for the case when a suitable local CLF is not found. Namely, the terminal set
in the exit-time optimal control problem is taken as a sufficiently small closed ball centered at the origin, the terminal cost is
chosen as zero, and we in particular establish sufficient conditions for the uniform convergence of the associated value function
to the original CLF from the infinite-horizon setting on compact subsets of the domain of asymptotic null-controllability as the
radius of the target ball tends to zero. Section 4 provides concluding remarks, and Appendix contains proofs of some auxiliary
statements. Subsequent practical developments are presented in the second part [1].

The following notation is adopted throughout the work:

- given integer numbers $j_1$ and $j_2 \geq j_1$, we write $i = j_1 / j_2$ instead of $i = j_1, j_1 + 1, \ldots, j_2$;
- the Minkowski sum of two sets $\Xi_1, \Xi_2$ in some linear space is defined as
  \[ \Xi_1 + \Xi_2 \overset{\text{def}}{=} \{ \xi_1 + \xi_2 : \xi_1 \in \Xi_1, \xi_2 \in \Xi_2 \}, \]
  and, if $\Xi_1 = \{ \xi \}$ is a singleton, we write $\xi + \Xi_2$ instead of $\{ \xi \} + \Xi_2$;
- given $j \in \mathbb{N}$ and $\Xi \subseteq \mathbb{R}^j$, the interior, closure, and boundary of $\Xi$ are denoted by $\text{int} \, \Xi, \bar{\Xi}$, and $\partial \Xi$, respectively;
- given $j \in \mathbb{N}$, the origin in $\mathbb{R}^j$ is written as $0_j$, $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^j$ (we avoid any confusions when considering
  the norms of vectors of different dimensions together), the open Euclidean ball with center $\xi \in \mathbb{R}^j$ and radius $r > 0$ is
denoted by $B_r(\xi)$, and its closure is $\overline{B_r(\xi)}$;
- given $j_1, j_2 \in \mathbb{N}$, the zero matrix of size $j_1 \times j_2$ is written as $0_{j_1 \times j_2}$, and the $j_1 \times j_1$ identity matrix is $I_{j_1 \times j_1}$;
- given $j \in \mathbb{N}$, a vector $\xi \in \mathbb{R}^j$ and a nonempty set $\Xi \subseteq \mathbb{R}^j$, the Euclidean distance from $\xi$ to $\Xi$ is denoted by $\text{dist} (\xi, \Xi)$;
- given $j_1, j_2 \in \mathbb{N}$, $\Xi_1 \subseteq \mathbb{R}^{j_1}$, and $\Xi_2 \subseteq \mathbb{R}^{j_2}$, the class of all essentially bounded functions $\varphi : \Xi_1 \to \Xi_2$ is denoted by
  $L_\infty (\Xi_1, \Xi_2)$, while $L_{\text{loc}}^\infty (\Xi_1, \Xi_2)$ is the wider class of all locally essentially bounded functions $\varphi : \Xi_1 \to \Xi_2$;
- given a function $\varphi : \Xi_1 \to \mathbb{R}$, the set of all its minimizers on $\Xi \subseteq \Xi_1$ is denoted by $\text{Arg min}_{\xi \in \Xi} \varphi (\xi)$, while the
  corresponding minimization problem is written as $\varphi (\xi) \to \inf_{\xi \in \Xi} \varphi (\xi)$ (or $\varphi (\xi) \to \min_{\xi \in \Xi}$ if the minimum exists);
- $\mathcal{K}$ is the class of all strictly increasing continuous functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying $\varphi (0) = 0$;
- $\mathcal{K}_\infty$ is the class of all functions $\varphi (\cdot) \in \mathcal{K}$ satisfying $\lim_{\rho \to +\infty} \varphi (\rho) = +\infty$;
- $\mathcal{L}$ is the class of all nonincreasing continuous functions $\varphi : [0, +\infty) \to [0, +\infty)$ for which $\lim_{\rho \to +\infty} \varphi (\rho) = 0$;
- $\mathcal{K}\mathcal{L}$ is the class of all continuous functions $\varphi : [0, +\infty)^2 \to [0, +\infty)$ such that $\varphi (\cdot, \rho) \in \mathcal{K}$ and $\varphi (\rho, \cdot) \in \mathcal{L}$ for every $\rho \geq 0$;
- if a vector variable $\xi$ consists of some arguments of a map $\varphi = \varphi (\ldots, \xi, \ldots)$, then $D_2 \varphi$ denotes the standard (Fréchet)
  partial derivative of $\varphi$ with respect to $\xi$, and $D \varphi$ is the standard derivative with respect to all of the vector arguments
  (the exact definitions of the derivatives depend on the domain and range of $\varphi$);
- given a real Hilbert space $X$, a nonempty set $\Xi \subseteq X$ and a point $\xi \in \Xi$, the proximal normal cone to $\Xi$ at $\xi$ is written as
  $N_p (\xi ; \Xi)$, and, if $\Xi$ is closed, $N (\xi ; \Xi)$ denotes the normal cone to $\Xi$ at $\xi$, which is polar to the related tangent cone
  $T(\xi ; \Xi)$ (see, e.g., \cite{60} §1.1, §2.5));
- given $j \in \mathbb{N}$, $\Xi \subseteq \mathbb{R}^j$, $\xi \in \text{int} \, \Xi, \xi \in \mathbb{R}^j$ and $\varphi : \Xi \to \mathbb{R}$, the lower Dini derivative (or the directional subderivate) of
  $\varphi$ at the point $\xi$ in the direction $\xi$ is written as $\partial^- \varphi (\xi ; \xi)$, the directional subdifferential (that is, the set of all directional
  subgradients) of $\varphi$ at $\xi$ is denoted by $D^- \varphi (\xi)$, and $D^+ \varphi (\xi)$ is the proximal subdifferential (that is, the set of all proximal
  subgradients) of $\varphi$ at $\xi$ (see, e.g., \cite{60} §0.1, §3.4)).

We also use the following definitions:

- given $j \in \mathbb{N}$ and a set $\Xi \subseteq \mathbb{R}^j$ containing the origin $0_j$, a function $\varphi : \Xi \to \mathbb{R} \cup \{ +\infty \}$ is called positive definite if
  $\varphi (0_j) = 0$ and $\varphi (\xi) > 0$ for all $\xi \in \Xi \setminus \{ 0_j \}$;
- given $j \in \mathbb{N}$ and a set $\Xi \subseteq \mathbb{R}^j$, a function $\varphi : \Xi \to \mathbb{R} \cup \{ -\infty, +\infty \}$ is called proper if the preimage $\varphi^{-1} (M) \subseteq \Xi$ of
  any compact set $M \subset \mathbb{R}$ is also compact.
2 GLOBAL EXTENSION OF A LOCAL CLF VIA EXIT-TIME OPTIMAL CONTROL

2.1 Problem statement and preliminary considerations

Let the state and control variables be denoted by $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, respectively. Consider the time-invariant system

$$
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)), \quad t \geq 0, \\
x(0) &= x_0 \in G, \\
u(\cdot) &\in U \overset{\text{def}}{=} L^\infty_\text{loc}([0, +\infty), U).
\end{align*}
$$

Assumption 2.1. The following conditions concerning (1) hold:

1) $U \subseteq \mathbb{R}^m$ is compact, $G \subseteq \mathbb{R}^n$ and $G_1 \subseteq \mathbb{R}^n$ are open domains, $\overline{G} \subset G_1$, and $0_n \in G$;

2) $G_1 \times U \ni (x, u) \mapsto f(x, u) \in \mathbb{R}^n$ is a continuous function;

3) any state trajectory of (1) defined on an interval $[0, T)$ with $T \in (0, +\infty) \cup \{+\infty\}$ and corresponding to $x_0 \in G$ and $u(\cdot) \in U$ stays inside $G$ and does not reach the boundary $\partial G$, that is, $G$ is a strongly invariant domain in the state space (see, e.g., [60], Chapter 4, §3) and note that $G = \mathbb{R}^n$ is a trivial case);

4) for any $R > 0$, there exists $C_{1,R} > 0$ satisfying

$$
\|f(x, u) - f(x', u)\| \leq C_{1,R} \|x - x'\| \quad \forall x, x' \in \overline{B}_R(0_n) \cap \overline{G} \quad \forall u \in U;
$$

5) there exist a continuously differentiable proper function $Y : G_1 \to [0, +\infty)$ and a continuous function $b : \mathbb{R} \to \mathbb{R}$ such that

$$
\sup_{u \in U} \langle DY(x), f(x, u) \rangle \leq b(Y(x)) \quad \forall x \in G_1
$$

and, for any $y_0 > 0$, the one-dimensional Cauchy problem $y(t) = b(y(t)), \ t \geq 0, \ y(0) = y_0$ admits an upper solution defined for all $t \geq 0$ (a typical case is $b(y) = C_2 y$ with a constant $C_2 > 0$).

Remark 2.2. For any $x_0 \in G$ and $u(\cdot) \in U$, let

$$
[0, T_{\text{ext}}(x_0, u(\cdot))] \ni t \mapsto x(t; x_0, u(\cdot)) \in G
$$

be a solution of the Cauchy problem (1) defined on the maximum extendability interval with the right endpoint $T_{\text{ext}}(x_0, u(\cdot)) \in (0, +\infty) \cup \{+\infty\}$. The local existence and uniqueness of the solutions follow from Items 1–4 of Assumption 2.1 while Item 5 is included in order to guarantee their extendability to the whole time interval $[0, +\infty)$. For verifying these implications, it suffices to recall basic results on Carathéodory ordinary differential equations [61], §1 and to note that Item 5 is related to the forward completeness property [62] and yields the boundedness of the reachable set

$$
\{x(t; x_0, u(\cdot)) : x_0 \in X_0, \ u(\cdot) \in U, \ t \in [0, \min \{T_{\text{ext}}(x_0, u(\cdot)), T\})
$$

for any finite time $T \in (0, +\infty)$ and any compact set $X_0 \subset G$ of initial states [63], Chapter 2, §2.1, Theorem 7). For example, if Items 1–3 of Assumption 2.1 hold and there exists a constant $C_1 > 0$ satisfying

$$
\|f(x, u) - f(x', u)\| \leq C_1 \|x - x'\| \quad \forall x, x' \in G \quad \forall u \in U,
$$

then Item 5 is fulfilled for $Y(x) = 1 + \|x\|^2$ and $b(y) = C_2 y$ with some constant $C_2 > 0$ (while Item 4 is a trivial corollary to (1)).

Items 1 and 2 of Assumption 2.1 ensure the compactness of the sets $\{f(x, u) : u \in U\}$ for all $x \in \overline{G}$. We also need their convexity.

Assumption 2.3. The set $\{f(x, u) : u \in U\}$ is convex for every $x \in \overline{G}$.

Now recall two underlying definitions (see, e.g., [5, 59]).

Definition 2.4. The region of asymptotic null-controllability for the system (1) (considered for $x \in G$) is given by

$$
D_0 \overset{\text{def}}{=} \left\{ x_0 \in G : \text{there exists } u(\cdot) \in U \text{ such that } \lim_{t \to +\infty} \|x(t; x_0, u(\cdot))\| = 0 \right\}.
$$
Definition 2.5. A continuous, proper and positive definite function \( V : D_0 \to [0, +\infty) \) is called a control Lyapunov function (CLF) for the system \( \{1\} \) in the region of asymptotic null-controllability \( D_0 \) if there exist a continuous and positive definite function \( W : D_0 \to [0, +\infty) \) and an open bounded domain \( \Theta \subset \mathbb{R}^n \) such that \( 0_\Theta \in \Theta, \, \Theta \subseteq D_0 \),

\[
\inf_{x \in D_0 \setminus \Theta} W(x) > 0,
\]

and the following infinitesimal decrease condition (involving lower Dini derivatives) holds:

\[
\inf_{u \in U} \partial^- V(x; f(x, u)) \leq -W(x) \quad \forall x \in D_0.
\]

Remark 2.6. Since it is possible that \( D_0 \neq \mathbb{R}^n \), the condition \([3]\) is included in the definition of a CLF so as to avoid the situation when \( \lim_{k \to \infty} W(x(k)) = 0 \) for a sequence \( \{x(k)\}_{k=1}^\infty \subseteq D_0 \) with \( \lim_{k \to \infty} x(k) \in \partial D_0 \). Note that \([5]\), Assumption (H3)] in particular serves to ensure this condition for the CLF characterization in \([5\), Theorem 4.4].

Remark 2.7. Let Items 1,2 of Assumption \([2.1] \) and Assumption \([2.3] \) hold. Suppose that \( E \subseteq G \) is an open domain, \( 0_n \in E \), and \( \Pi_i : E \to \mathbb{R}, \ i = 1, 2, \) are continuous and positive definite functions. At a state \( x \in E \), consider the infinitesimal decrease conditions

\[
\inf_{u \in U} \partial^- \Pi_i(x; f(x, u)) \leq -\Pi_2(x), \quad (5)
\]

\[
\max_{u \in U} \{ -\langle \zeta, f(x, u) \rangle \} \geq \Pi_2(x) \quad \forall \zeta \in D^+ \Pi_i(x), \quad (6)
\]

\[
\max_{u \in U} \{ -\langle \zeta, f(x, u) \rangle \} \geq \Pi_2(x) \quad \forall \zeta \in D \Pi_i(x), \quad (7)
\]
in the Dini, proximal and viscosity forms, respectively. If \([5]\) holds at a state \( x \in E \), then \([6]\) and \([7]\) also hold at this state (see \([60\), pp. 136, 138]). Furthermore, the following three statements are equivalent (see \([59\), p. 27], \([60\), Chapter 3, Theorem 4.2] and \([64\), Theorem 9.2]): (i) \([5]\) holds for all \( x \in E \); (ii) \([6]\) holds for all \( x \in E \); (iii) \([7]\) holds for all \( x \in E \). Thus, the Dini, proximal and viscosity decrease conditions lead to equivalent definitions of a CLF.

The next assumption plays a significant role and states the existence of a function that locally satisfies the CLF conditions and some other technical properties.

Assumption 2.8. The following conditions hold:

1) \( \Omega \subseteq G \) is an open domain, and \( 0_n \in \Omega \);

2) \( V_{\text{loc}} : \overline{\Omega} \to [0, +\infty) \) is a continuous, proper and positive definite function, whose restriction to \( \Omega \) satisfies the infinitesimal decrease condition

\[
\inf_{u \in U} \partial^- V_{\text{loc}}(x; f(x, u)) \leq -W_{\text{loc}}(x) \quad \forall x \in \Omega
\]

with some continuous and positive definite function \( W_{\text{loc}} : \Omega \to [0, +\infty) \);

3) \( V_{\text{loc}}(\cdot) \) is locally Lipschitz continuous in \( \Omega \) (and hence Lipschitz continuous on any compact subset of \( \Omega \) \([65\), Theorem 1.14]);

4) there exist positive constants \( c \) and \( C_3 \) such that the set \( \{ x \in \overline{\Omega} : V_{\text{loc}}(x) < c \} \) is a bounded open domain in \( \mathbb{R}^n \), whose closure coincides with the set

\[
\Omega_c \overset{\text{def}}{=} \left\{ x \in \overline{\Omega} : V_{\text{loc}}(x) \leq c \right\}
\]

and fulfills the inclusion

\[
\Omega_c + B_{C_3}(0_n) \subseteq \Omega,
\]

while the boundary \( \partial \Omega_c \) coincides with

\[
l_c \overset{\text{def}}{=} \left\{ x \in \overline{\Omega} : V_{\text{loc}}(x) = c \right\}
\]

and is a connected piecewise regular hypersurface in \( \mathbb{R}^n \);

5) \( \lim_{\|x\| \to 0} \sup \left\{ \|x\| : x \in \overline{\Omega}, \ V_{\text{loc}}(x) \leq \varepsilon \right\} = 0. \)

Remark 2.9. Due to Remark \([2.7]\) the condition \([8]\) in Item 2 of Assumption \([2.8]\) can also be written in the proximal and viscosity forms.
**Lemma 2.13.** The proof of Proposition 2.12 requires two auxiliary results from nonsmooth analysis. The proof of the first of them (Lemma 2.13) is rather straightforward and given in Appendix, while the proof of the second result (Lemma 2.14) is essentially more difficult and can be found in [67].

**Lemma 2.13.** If $E \subseteq \mathbb{R}^n$ is an open set and a function $\varphi : E \to \mathbb{R}$ is Lipschitz continuous with constant $C > 0$, then

$$\|\xi\| \leq C\sqrt{n} \quad \forall \xi \in D_p^\text{loc}\varphi(x) \quad \forall x \in E. \tag{13}$$

**Lemma 2.14.** Assume that $X$ is a real Hilbert space, $\varphi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a proper and lower semicontinuous function, $M \overset{\text{def}}{=} \{\xi \in X : \varphi(\xi) \leq 0\}$, $x \in M$, and $v \in N_p(x; M)$. Then at least one of the following two properties holds:

1) for any $\epsilon > 0$, there exist $x' \in X$ and $\xi' \in D_p^\text{loc}\varphi(x')$ such that

$$\|x' - x\| < \epsilon, \quad \|\varphi(x') - \varphi(x)\| < \epsilon, \quad \|\xi'\| < \epsilon;$$

2) for any $\epsilon > 0$, there exist $x' \in X$, $\xi' \in D_p^\text{loc}\varphi(x')$ and $\lambda > 0$ such that

$$\|x' - x\| < \epsilon, \quad \|\varphi(x') - \varphi(x)\| < \epsilon, \quad \|v - \lambda\xi'\| < \epsilon.$$

**Proof of Proposition 2.12** Since $l_c$ is compact and $V_{\text{loc}}(\cdot)$ are continuous and positive definite, there exist constants $\eta_1 > 0$, $\eta_2 \in (0, C_3)$ ($C_3$ was introduced in Item 4 of Assumption 2.8), $\tau_3 > 0$ such that $B_{\eta_1}(0_n) \subseteq \Omega$ and

$$W_{\text{loc}}(x) \geq \eta_3 \quad \forall x \in l_c + B_{\eta_2}(0_n). \tag{14}$$

In line with Remark 2.9, the infinitesimal decrease condition on $V_{\text{loc}}(\cdot)$ can be written in the proximal form:

$$\min_{u \in U} \langle \xi, f(x, u) \rangle \leq -W_{\text{loc}}(x) \quad \forall \xi \in D_p^\text{loc}V_{\text{loc}}(x) \quad \forall x \in \Omega. \tag{15}$$

From the relations (14), (15), (11) and $\eta_2 \in (0, C_3)$, one obtains

$$\min_{u \in U} \langle \xi, f(x, u) \rangle \leq -\eta_3 \quad \forall \xi \in D_p^\text{loc}V_{\text{loc}}(x) \quad \forall x \in l_c + B_{\eta_2}(0_n). \tag{16}$$

The property (14), continuity of $f(\cdot, \cdot)$, and compactness of $U$ and $l_c + B_{\eta_2}(0_n)$ yield the existence of a constant $\eta_4 > 0$ satisfying

$$\|\xi\| \geq \eta_4 \quad \forall \xi \in D_p^\text{loc}V_{\text{loc}}(x) \quad \forall x \in l_c + B_{\eta_2}(0_n). \tag{17}$$

Moreover, the Lipschitz continuity of $V_{\text{loc}}(\cdot)$ on compact subsets of $\Omega$ and Lemma 2.13 guarantee the existence of a constant $\eta_5 > 0$ such that

$$\|\xi\| \leq \eta_5 \quad \forall \xi \in D_p^\text{loc}V_{\text{loc}}(x) \quad \forall x \in l_c + B_{\eta_2}(0_n). \tag{18}$$

Now let us apply Lemma 2.14 to the zero sublevel set $\Omega_c$ of the proper and lower semicontinuous function that equals $V_{\text{loc}}(x) - c$ for $x \in \Omega$ and $+\infty$ for $x \in \mathbb{R}^n \setminus \Omega$.

Take $x \in l_c$ and $v \in N_p(x; \Omega_c)$ with $\|v\| = 1$. 
By assuming the optimal control problem, stated with respect to the target set \( \Omega \), \( V(x) < c \) in \( \Omega \), \( V(x) > c \) outside \( \Omega \), \( V(x) = V_{loc}(x) = c \) on \( l_c \).

\[ V_{loc}(x) < c \text{ in } \Omega_c, \quad V_{loc}(x) > c \text{ outside } \Omega_c, \quad V(x) = V_{loc}(x) = c \text{ on } l_c. \]

\[ x(T_c(x_0, u()); x_0, u()) \in l_c \]

**FIGURE 1** The exit-time optimal control problem (23), whose target set is a level set of a local CLF.

By virtue of (17), Item 1 of Lemma 2.14 does not hold in the considered situation. Then Item 2 of Lemma 2.14 holds and implies that, for any \( \epsilon > 0 \), there exist \( x' \in B_r(x) \), \( \zeta' \in D_p V_{loc}(x') \) and \( \lambda > 0 \) satisfying

\[ \|v - \lambda \zeta'\| < \epsilon. \]  

(19)

By assuming \( \epsilon \in (0, 1) \) without loss of generality, and by using (19) with \( \|v\| = 1 \), it is easy to derive \( |\lambda\|\zeta'\| - 1| < \epsilon, \|\zeta'\| > 0 \), and therefore

\[ \left\|\lambda \zeta' - \left(1 - \frac{1}{\|\zeta'\|}\right) \|\zeta'\| \right\| = |\lambda - 1 - \frac{1}{\|\zeta'\|}| \|\zeta'\| = |\lambda\|\zeta'\| - 1| < \epsilon. \]  

(20)

The inequalities (19) and (20) lead to

\[ \|v - \zeta'\| < 2\epsilon. \]  

(21)

Thus, for any \( \epsilon > 0 \), there exist \( x' \in B_r(x) \) and \( \zeta' \in D_p V_{loc}(x') \) such that (21) holds. Together with the relations (16), (18) and continuity of the function \( \mathbb{R}^n \times G \ni (\xi_1, \xi_2) \mapsto \min_{u \in U} \langle \xi_1, f(\xi_2, u) \rangle \), this ensures that, for any \( \epsilon > 0 \), there exist \( x' \in B_{r_2}(x) \) and \( \zeta' \in D_p V_{loc}(x') \) for which

\[ \min_{u \in U} \langle v, f(x, u) \rangle \leq \min_{u \in U} \left\langle \frac{\zeta'}{\|\zeta'\|}, f(x', u) \right\rangle + \epsilon \leq -\frac{\eta_3}{\|\zeta'\|} + \epsilon \leq -\frac{\eta_3}{\eta_5} + \epsilon. \]

Since \( \epsilon > 0 \) can be taken arbitrarily small, the Petrov condition (12) holds with \( C_4 = \eta_3/\eta_5 \).

(\( \square \))

Other important properties are the openness, connectedness and weak invariance of the region of asymptotic null-controllability (recall Definition 2.4).

**Proposition 2.15.** Under Assumptions 2.1, 2.3 and 2.8, \( D_0 \) is an open domain (that is, an open connected set) containing \( \Omega_c \), and it is weakly invariant in the sense that, for any \( x_0 \in D_0 \), there exists \( u(\cdot; x_0) \in \mathcal{U} \) satisfying \( x(t; x_0, u(\cdot; x_0)) \in D_0 \) for all \( t \geq 0 \).

**Proof.** The inclusion \( \Omega_c \subseteq D_0 \) was justified in Remark 2.10. The connectedness and weak invariance of \( D_0 \) can be established similarly to [5], Proposition 2.3, (iii). By using Proposition 2.12, 66, Remark 8.1.6, and the reasonings in 66, the proofs of Theorems 8.2.1 and 8.2.3, one can show that \( D_0 \) is open. In [66, Chapter 8], the global Lipschitz condition is imposed on \( f(\cdot, u) \) uniformly with respect to \( u \in \mathcal{U} \), but it can in fact be relaxed to Items 4 and 5 of Assumption 2.1 when verifying the openness of \( D_0 \).

(\( \square \))

Next, let us adopt the convention \( \inf \emptyset = +\infty \) and introduce the minimum times of reaching \( \Omega_c \):

\[ T_{\Omega_c}(x_0, u(\cdot)) = \inf \{ T \in [0, +\infty) : x(T; x_0, u(\cdot)) \in \Omega_c \} \quad \forall x_0 \in G \quad \forall u(\cdot) \in \mathcal{U}. \]  

(22)

A key point of this section is to represent a sought-after CLF outside the sublevel set \( \Omega_c \) as the value function in an exit-time optimal control problem, stated with respect to the target set \( l_c \) and the constant terminal cost \( V_{loc}(x) = c \) for \( x \in l_c \) (see Fig. 1):

\[ V(x_0) = \inf_{u(\cdot) \in \mathcal{U} : T_{\Omega_c}(x_0, u(\cdot)) < +\infty} \left\{ T_{\Omega_c}(x_0, u(\cdot)) \right\} + \int_0^{T_{\Omega_c}(x_0, u(\cdot))} g(x(t; x_0, u(\cdot)), u(t)) \, dt + c \]  

\forall x_0 \in G \setminus \Omega_c. \]  

(23)
In order to characterize the domain of asymptotic null-controllability $D_0$ via the value function $V(\cdot)$, a number of conditions are imposed on the running cost.

**Assumption 2.16.** The following conditions concerning the running cost $g(\cdot, \cdot)$ hold:

1) $\overline{G} \times U \ni (x, u) \mapsto g(x, u) \in [0, +\infty)$ is a nonnegative continuous function;

2) for any $R > 0$, there exists $C_{5,R} > 0$ such that
   \[|g(x, u) - g(x', u)| \leq C_{5,R} \|x - x'\| \quad \forall x, x' \in \overline{B}_R(0) \cap \overline{G} \quad \forall u \in U;\] (24)

3) $g(x, u) > 0$ for all $x \in \overline{G} \setminus \{0_n\}$ and $u \in U$;

4) $C_6 \overset{\text{def}}{=} \inf \{g(x, u) : x \in \overline{G} \setminus (\text{int}\Omega_c), u \in U\} > 0$.

**Proposition 2.17.** Under Assumptions 2.1, 2.3, 2.8 and 2.16, the following relations hold for the value function (23):

\[V(x_0) = +\infty \quad \forall x_0 \in G \setminus D_0,\] (25)

\[V(x_0) > c \quad \forall x_0 \in D_0 \setminus \Omega_c.\] (26)

**Proof.** The property (25) is clear due to the definition (23) and Proposition 2.15. For establishing (26), let us take $x_0 \in D_0 \setminus \Omega_c$ and show that

\[T_{\Omega_c}(x_0) \overset{\text{def}}{=} \inf_{u(\cdot) \in U} T_{\Omega_c}(x_0, u(\cdot)) > 0.\] (27)

Assume $T_{\Omega_c}(x_0) = 0$. Then there exist a number $T > 0$ and a sequence $\{u^{(k)}(\cdot)\}^\infty_{k=1} \subset U$ such that $T^{(k)} \overset{\text{def}}{=} T_{\Omega_c}(x_0, u^{(k)}(\cdot)) \leq T$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} T^{(k)} = 0$. According to Remark 2.2, the reachable set

\[X_T(x_0) \overset{\text{def}}{=} \{x(t ; x_0, u(\cdot)) : t \in [0,T], u(\cdot) \in U\} \subset \overline{G}
\]

is bounded. Since $f(\cdot, \cdot)$ is continuous and $U$ is compact, one has

\[M_T(x_0) \overset{\text{def}}{=} \max_{x \in X_T(x_0), u \in U} \|f(x, u)\| < +\infty.\]

Hence,

\[0 \leq \lim_{k \to \infty} \|x(T^{(k)}; x_0, u^{(k)}(\cdot)) - x_0\| \leq M_T(x_0) \lim_{k \to \infty} T^{(k)} = 0,
\]

which contradicts with $x_0 \notin \Omega_c$. This implies (27). From (23), (27) and Item 4 of Assumption 2.16 one obtains $V(x_0) - c \geq C_6 T_{\Omega_c}(x_0) > 0$. \qed

It is reasonable to extend the function (23) to $\Omega_c$ by

\[V(x_0) \overset{\text{def}}{=} V_{\text{loc}}(x_0) \quad \forall x_0 \in \Omega_c\] (28)

(see Fig. 1).

**Proposition 2.18.** Let Assumptions 2.1, 2.3, 2.8 and 2.16 hold, and consider the function $V(\cdot)$ defined by (23) and (28). The following properties hold:

\[V(x_0) < +\infty \quad \forall x_0 \in D_0,\]

\[V(x_0) = +\infty \quad \forall x_0 \in G \setminus D_0,\]

\[V(x_0) = V_{\text{loc}}(x_0) < c \quad \forall x_0 \in \text{int}\Omega_c,\]

\[V(x_0) = V_{\text{loc}}(x_0) = c \quad \forall x_0 \in l_c = \partial\Omega_c,\]

\[V(x_0) > c \quad \forall x_0 \in G \setminus \Omega_c,\]

\[V_{\text{loc}}(x_0) > c \quad \forall x_0 \in \Omega \setminus \Omega_c.\]

**Proof.** These relations can be directly obtained by using Definition 2.3, Assumption 2.8, Remark 2.10 and Proposition 2.17. \qed
One more technical assumption will be required below to prove the properness of $V(\cdot)$.

**Assumption 2.19.** There exist positive constants $C_7, C_8$ such that

$$g(x,u) \geq C_8 \| f(x,u) \| \quad \forall x \in G \setminus B_{C_7}(0_n) \quad \forall u \in U.$$ 

## 2.2 Main result

The main result of this section (Theorem 2.21) indicates that, under the adopted assumptions, the concatenation of the local CLF in $\Omega$, with the value function for the exit-time optimal control problem (23) is a global CLF in the whole domain of asymptotic null-controllability. Before verifying the main result, let us establish some auxiliary properties.

**Proposition 2.20.** Under Assumptions 2.1, 2.3, 2.8, 2.16 and 2.19 the following properties hold for the function $V(\cdot)$ defined by (23) and (28):

1) $V(\cdot)$ is locally Lipschitz continuous in $D_0$;

2) the restriction of $V(\cdot)$ to $D_0 \setminus \Omega_c$ solves the HJB equation

$$\max_{u \in U} \{ -\langle DV(x), f(x,u) \rangle - g(x,u) \} = 0, \quad x \in D_0 \setminus \Omega_c,$$

in the viscosity sense;

3) for any sequence $\{x^{(k)}\}_{k=1}^{\infty} \subset D_0$ satisfying either $\lim_{k \to \infty} x^{(k)} = x' \in \partial D_0$ or $\lim_{k \to \infty} \|x^{(k)}\| = +\infty$, one has $\lim_{k \to \infty} V(x^{(k)}) = +\infty$.

**Proof.** For verifying Items 1, 2, as well as Item 3 for $\lim_{k \to \infty} x^{(k)} = x' \in \partial D_0$, it suffices to use [66, Remark 8.1.6] and the reasonings in [66] the proofs of Theorem 8.2.5, Theorem 8.1.8 and Proposition 8.2.6. As in the proof of Proposition 2.15, Items 4 and 5 of Assumption 2.1 replace the requirement that $f(\cdot,u)$ satisfy the global Lipschitz condition uniformly with respect to $u \in U$.

It remains to prove Item 3 in case $\lim_{k \to \infty} \|x^{(k)}\| = +\infty$. Consider such a sequence $\{x^{(k)}\}_{k=1}^{\infty} \subset D_0$. In line with Assumption 2.19 and the compactness of $\Omega_c$, there exists a constant $C_7 \geq C_7$ satisfying

$$\Omega_c \subseteq B_{C_7}(0_n),$$

$$g(x,u) \geq C_8 \| f(x,u) \| \quad \forall x \in G \setminus B_{C_7}(0_n) \quad \forall u \in U.$$

Denote

$$T_k(u(\cdot)) \overset{\text{def}}{=} \inf \left\{ T \in [0, +\infty) : x(T; x^{(k)}, u(\cdot)) \in \overline{B}_{C_7}(0_n) \right\} \quad \forall u(\cdot) \in U' \quad \forall k \in \mathbb{N}.$$ 

Then one has

$$V(x^{(k)}) \geq \inf_{u(\cdot) \in U'} \left\{ \int_0^{T_k(u(\cdot))} g \left( x \left( t; x^{(k)}, u(\cdot) \right), u(t) \right) \, dt \right\}$$

$$\geq C_8 \inf_{u(\cdot) \in U'} \left\{ \int_0^{T_k(u(\cdot))} \| f \left( x \left( t; x^{(k)}, u(\cdot) \right), u(t) \right) \| \, dt \right\}$$

$$\geq C_8 \inf_{u(\cdot) \in U'} \left\{ \left\| x \left( T_k(u(\cdot)); x^{(k)}, u(\cdot) \right) - x^{(k)} \right\| \right\}$$

$$\geq C_8 \left( \left\| x^{(k)} \right\| - C_7 \right)$$

defined

for all $k \in \mathbb{N}$. Together with $\lim_{k \to \infty} \|x^{(k)}\| = +\infty$, this leads to $\lim_{k \to \infty} V(x^{(k)}) = +\infty$.

**Theorem 2.21.** Let Assumptions 2.1, 2.3, 2.8, 2.16 and 2.19 hold. The function $V(\cdot)$ defined by (23) and (28) is a CLF for the system (1) in $D_0$, i.e., the restriction of this function to $D_0$ is a continuous, proper, positive definite and such that the infinitesimal decrease condition

$$\inf_{u \in U} \partial^- V(x; f(x,u)) \leq -W(x) \quad \forall x \in D_0$$

(29)
holds with some continuous and positive definite function \( W : D_0 \to [0, +\infty) \) satisfying
\[
\inf_{x \in G \setminus (\text{int } \Omega_c)} W(x) > 0. \tag{30}
\]
Furthermore, \( V(\cdot) \) is locally Lipschitz continuous in \( D_0 \) and therefore differentiable almost everywhere in \( D_0 \) (with respect to the Lebesgue measure in \( \mathbb{R}^n \)).

Proof. In line with Item 1 of Proposition \ref{prop:2.20}, \( V(\cdot) \) is locally Lipschitz continuous in \( D_0 \), and it is differentiable almost everywhere in \( D_0 \) due to Rademacher’s theorem. The positive definiteness of \( V(\cdot) \) directly follows from Proposition \ref{prop:2.17} and the positive definiteness of \( V_{\text{loc}}(\cdot) \).

Let us show that \( V(\cdot) \) is proper. According to the relation \ref{eq:25} and Item 3 of Proposition \ref{prop:2.20}, it suffices to verify the properness of the restriction of \( V(\cdot) \) to \( D_0 \). The continuity of the latter implies that the preimages of closed sets are closed. Again due to Item 3 of Proposition \ref{prop:2.20}, the considered restriction is also such that the preimages of bounded sets are bounded. One consequently obtains the compactness of the preimages of compact sets, which means properness.

It remains to establish the infinitesimal decrease condition \ref{eq:29} with an appropriate function \( W(\cdot) \).

Since the function \( W_{\text{loc}} : \Omega \to [0, +\infty) \) (introduced in Item 2 of Assumption \ref{assump:2.16}) is continuous and the set \( \Omega_c \subset \Omega \) is compact, Tietze’s extension theorem (see, e.g., \cite[Theorem 5.2.1]{68}) ensures the existence of a continuous function \( W_1 : \mathbb{R}^n \to \mathbb{R} \) satisfying \( W_1(x) = W_{\text{loc}}(x) \) for all \( x \in \Omega_c \). Bearing in mind also the positive definiteness of \( W_{\text{loc}}(\cdot) \) and the compactness of the boundary \( I_c = \partial \Omega_c \) that does not contain \( 0_n \), one concludes
\[
\min_{x \in I_c} W_{\text{loc}}(x) > 0. \tag{31}
\]
Hence, the function
\[
W_2(x) \overset{\text{def}}{=} \begin{cases} W_{\text{loc}}(x), & x \in \Omega_c, \\ \max \left\{ W_1(x), \min_{\xi \in I_c} W_{\text{loc}}(\xi) \right\}, & x \in \mathbb{R}^n \setminus \Omega_c, \end{cases} \tag{32}
\]
is continuous and positive definite. Now take
\[
W(x) \overset{\text{def}}{=} \min_{u \in \mathbb{U}} \left\{ W_2(x), \min_{u \in \mathbb{U}} g(x, u) \right\} \quad \forall x \in D_0. \tag{33}
\]

The compactness of \( \mathbb{U} \) and Items 1, 3 of Assumption \ref{assump:2.16} yield that the function \( G \ni x \mapsto \min_{u \in \mathbb{U}} g(x, u) \) is continuous everywhere in \( \mathbb{G} \) and positive for all \( x \in \mathbb{G} \setminus \{ 0_n \} \). Thus, \( \mathbb{W} \) is a continuous and positive definite function. Moreover, it satisfies \ref{eq:30} due to \ref{eq:31} and Item 4 of Assumption \ref{assump:2.16}.

In order to establish the property \ref{eq:29} with the selected \( W(\cdot) \), it suffices to verify this Dini form of the infinitesimal decrease condition for \( x \in \Omega_c \) and the related viscosity form for \( x \in D_0 \setminus \Omega_c \) (recall Remark \ref{rem:2.7}).

For \( x \in \text{int } \Omega_c \), the inequality in \ref{eq:29} holds due to Assumption \ref{assump:2.8} For \( x \in D_0 \setminus \Omega_c \), Item 2 of Proposition \ref{prop:2.20} implies the viscosity form of the infinitesimal decrease condition:
\[
\max_{u \in \mathbb{U}} \{ -\langle \zeta, f(x, u) \rangle \} - W(x) \geq \max_{u \in \mathbb{U}} \{ -\langle \zeta, f(x, u) \rangle \} - \min_{u \in \mathbb{U}} g(x, u) \geq \max_{u \in \mathbb{U}} \{ -\langle \zeta, f(x, u) \rangle - g(x, u) \} \geq 0
\]
\( \forall \zeta \in D^- V(x) \).

It therefore remains to prove the inequality in \ref{eq:29} for \( x \in I_c = \partial \Omega_c \).

Let \( x \in I_c \). Due to the local Lipschitz continuity of \( V(\cdot) \) and \( V_{\text{loc}}(\cdot) \) in \( D_0 \) and \( \Omega_c \), respectively, the following representations for the lower Dini derivatives hold (see, e.g., \cite[Remark 3.1.4]{66}):
\[
\partial^- V(x; \zeta) = \liminf_{\lambda \to 0^+} \frac{V(x + \lambda \zeta) - V(x)}{\lambda}, \\
\partial^- V_{\text{loc}}(x; \zeta) = \liminf_{\lambda \to 0^+} \frac{V_{\text{loc}}(x + \lambda \zeta) - V_{\text{loc}}(x)}{\lambda} \tag{34}
\]
\( \forall \zeta \in \mathbb{R}^n \).
Introduce the control subset
\[ U_x \overset{\text{def}}{=} \{ u \in U : \text{ there exists a sequence } \{ \lambda_k \}_{k=1}^{\infty} \subset (0, +\infty) \text{ such that} \]
\[ \lim_{k \to \infty} \lambda_k = 0 \text{ and } x + \lambda_k f(x, u) \in \Omega_c \text{ for all } k \in \mathbb{N} \}, \]
which is nonempty by virtue of Proposition 2.12. With the help of Proposition 2.18 and the property (34) (see also Fig. [1], one obtains
\[ \inf_{u \in U_x} \partial^{-} V(x; f(x, u)) \leq 0, \quad \inf_{u \in U \setminus U_x} \partial^{-} V(x; f(x, u)) \geq 0, \]
\[ \inf_{u \in U_x} \partial^{-} V_{loc}(x; f(x, u)) \leq 0, \quad \inf_{u \in U \setminus U_x} \partial^{-} V_{loc}(x; f(x, u)) \geq 0, \]
and
\[ \inf_{u \in U_x} \partial^{-} V(x; f(x, u)) = \inf_{u \in U_x} \partial^{-} V_{loc}(x; f(x, u)) = \inf_{u \in U} \partial^{-} V_{loc}(x; f(x, u)) \]
(the relations between \( V(\cdot), V_{loc}(\cdot) \) and \( c \) in Proposition 2.18 help to determine the signs of the numerators in (34)). Together with (8), (32) and (33), this leads to
\[ \inf_{u \in U} \partial^{-} V(x; f(x, u)) = \inf_{u \in U} \partial^{-} V_{loc}(x; f(x, u)) \leq -W_{loc}(x) \leq -W(x) \]
and thereby completes the proof. \( \square \)

2.3 | Investigation of the exit-time optimal control problem

As was shown in the previous subsection, if one can find a suitable local CLF \( V_{loc}(\cdot) \) and the conditions of Theorem 2.21 are fulfilled, the value function in the exit-time optimal control problem (23) extends the local CLF outside the sublevel set \( x_0 \in D_0 \setminus \Omega_c \), so that the resulting function \( V(\cdot) \) becomes a global CLF in the whole domain of asymptotic null-controllability \( D_0 \).

In order to verify the existence of optimal control strategies and to use necessary optimality conditions (Pontryagin’s principle) for the exit-time problem (23) with \( x_0 \in D_0 \setminus \Omega_c \), let us reformulate it as

\[ V(x_0) \overset{\text{def}}{=} \inf_{u(\cdot) \in U_0; T \in [0, +\infty)} \left\{ \int_{0}^{T} g(x(t; x_0, u_t)), u(t) \text{ dr} + c \right\} \quad \forall x_0 \in G \setminus \Omega_c. \]

(35)

It is easy to see that (23) and (35) are equivalent under Assumptions 2.1, 2.3, 2.8 and 2.16. Some additional conditions also need to be imposed.

Assumption 2.22. The set
\[ \{(f(x, u), y) \in \mathbb{R}^n \times \mathbb{R} : u \in U, \ y \geq g(x, u)\} \]
is convex for every \( x \in \overline{G} \).

Remark 2.23. Assumption 2.22 strengthens Assumption 2.3. One can easily verify that a sufficient condition for the fulfillment of Assumption 2.22 is the convexity of the set
\[ \{(f(x, u), g(x, u)) \in \mathbb{R}^n \times \mathbb{R} : u \in U\} \]
for all \( x \in \overline{G} \). \( \square \)

Assumption 2.24. The functions \( G \ni x \mapsto f(x, u) \in \mathbb{R}^n \) and \( G \ni x \mapsto g(x, u) \in [0, +\infty) \) are continuously differentiable for every \( u \in U \).

Theorem 2.25. Let Assumptions 2.1, 2.8, 2.16 and 2.22 hold. For any fixed initial state \( x_0 \in D_0 \setminus \Omega_c \), there exists an optimal control strategy for the exit-time problem (23) or, equivalently, for (35).

The proof of Theorem 2.25 uses the general existence result from [41, §9.3] and is provided in Appendix.

Theorem 2.26. (Pontryagin’s principle; see, e.g., [41, §5.1, §4.2 (emphasize Remark 10), §4.4.B], [42, §2.4]) Let Assumptions 2.1, 2.8, 2.16, 2.22 and 2.24 hold. Consider an optimal control strategy \( u^*(\cdot) \in U \) in the exit-time problem (23) or,
equivalently, in (35) for a fixed initial state \( x_0 \in D_0 \setminus \Omega_c \). Denote \( T^* \overset{\text{def}}{=} T_\Omega (x_0, u^*(\cdot)) < +\infty \), and let

\[
[0, T^*] \ni t \mapsto x^*(t) = x(t; x_0, u^*(\cdot)) \in G
\]

be the corresponding optimal state trajectory. Moreover, introduce the Hamiltonian:

\[
H(x, u, p, \bar{p}) \overset{\text{def}}{=} \langle p, f(x, u) \rangle + \bar{p} g(x, u),
\]

(36)

\[
H(x, p, \bar{p}) \overset{\text{def}}{=} \min_{u' \in U} H(x, u', p, \bar{p})
\]

\( \forall (x, u, p, \bar{p}) \in G \times U \times \mathbb{R}^n \times \mathbb{R} \).

Then there exist a function \( p^*: [0, T^*] \to \mathbb{R}^n \) and a constant \( \bar{p}^* \geq 0 \) such that the following properties hold:

\begin{itemize}
  \item \( (p^*(t), \bar{p}^*) \neq 0_{n+1} \) for every \( t \in [0, T^*] \);
  \item \( (x^*(\cdot), p^*(\cdot)) \) is an absolutely continuous solution of the characteristic boundary value problem
    \[
    \begin{aligned}
    \dot{x}^*(t) &= D_p H(x^*(t), u^*(t), p^*(t), \bar{p}^*) = f(x^*(t), u^*(t)), \\
    \dot{p}^*(t) &= -D_x H(x^*(t), u^*(t), p^*(t), \bar{p}^*) \\
    &= -D_x f(x^*(t), u^*(t))^\top p^*(t) - \bar{p}^* D_x g(x^*(t), u^*(t)),
    \end{aligned}
    \]
    (t \in [0, T^*]),
    \]
    \[
    x^*(0) = x_0,
    \]
    \[
    x^*(T^*) \in L_c, \quad p^*(T^*) \in N(x^*(T^*); \Omega_c)
    \]
    (the notation for normal cones was described in the introduction);
  \item the Hamiltonian minimum condition
    \[
    H(x^*(t), u^*(t), p^*(t), \bar{p}^*) = H(x^*(t), p^*(t), \bar{p}^*)
    \]
    is satisfied for almost all \( t \in [0, T^*] \) (with respect to the Lebesgue measure in \( \mathbb{R} \));
  \item the Hamiltonian vanishes along the optimal characteristic trajectory, i.e.,
    \[
    H(x^*(t), p^*(t), \bar{p}^*) \equiv 0 \quad \forall t \in [0, T^*].
    \]
    (39)
\end{itemize}

**Remark 2.27.** Since the Hamiltonian (36) is positive homogeneous of degree 1 with respect to \( (p, \bar{p}) \), it suffices to consider only the two cases \( \bar{p}^* = 0 \) and \( \bar{p}^* = 1 \) in Theorem 2.26. The case \( \bar{p}^* = 0 \) is called abnormal.

For handling the infinite value \(+\infty\), consider the Kruzhkov transformed function

\[
\nu(x_0) \overset{\text{def}}{=} 1 - e^{-V(x_0)} \in [0, 1] \quad \forall x_0 \in G
\]

(40)

with the convention \( e^{-(+\infty)} \overset{\text{def}}{=} 0 \). Note that the function \( \mathbb{R} \ni \xi \mapsto 1 - e^{-\xi} \) vanishes at \( \xi = 0 \), tends to 1 as \( \xi \to +\infty \), strictly increases, and is infinitely differentiable.

**Theorem 2.28.** Let Assumptions 2.1, 2.3, 2.8 and 2.16 hold, and consider the functions \( V(\cdot), \nu(\cdot) \) defined by (23), (28), (40).

The domain of asymptotic null-controllability can be represented as

\[
D_0 = \{ x_0 \in G : V(x_0) < +\infty \} = \{ x_0 \in G : \nu(x_0) < 1 \}.
\]

**Proof.** It suffices to recall Proposition 2.18.

Introduce also the set-valued extremal control map:

\[
U^*(x, p, \bar{p}) \overset{\text{def}}{=} \text{Arg min}_{u \in U} H(x, u, p, \bar{p}) \quad \forall (x, p, \bar{p}) \in G \times \mathbb{R}^n \times \mathbb{R}.
\]

(41)

As was discussed in (34), (57), (58), characteristic boundary value problems, such as (37), may admit multiple solutions, some of which may not be optimal, and it is therefore relevant to parametrize the characteristic fields with respect to the extended initial costate \((p_0, \bar{p}^*)\) in case of (37) and to solve the related Cauchy problems. Solutions of the latter are unique if, for example, the
Moreover, the same value is obtained when minimizing over the bounded set for all extended initial costates (44).

Moreover, the same value is obtained when minimizing over the bounded set

and even over its subset

Proof. The first statement directly follows from Theorems 2.25, 2.26, Remark 2.27 and the fact that, compared to the boundary value problems (37), (38), the Cauchy problems (43), (44) generate a wider characteristic field (due to the absence of the transversality condition on the terminal costate).

Since the Hamiltonian is positive homogeneous of degree 1 with respect to \((p, \bar{p})\), the extremal control map (41) satisfies

and the state components of the characteristic trajectories do not change after multiplying \((p_0, \bar{p}^\ast)\) by any positive number. Together with the Hamiltonian vanishing condition (39) in Theorem 2.26, this yields the second statement.

Besides, let us separately formulate the well-known Hamiltonian conservation property as applied to (43). For convenience, its proof is given in Appendix.

Proposition 2.30. Under the conditions of Theorem 2.29, the Hamiltonian is conserved along any solution of the characteristic Cauchy problem (43) with \((x_0, p_0, \bar{p}^\ast) \in G \times \mathbb{R}^n \times [0, +\infty)\).

The next section modifies the theoretical constructions of the current section for the case when an appropriate local CLF is not available.
3 | APPROXIMATION OF A GLOBAL CLF WHEN AN APPROPRIATE LOCAL CLF IS NOT AVAILABLE

The results of the previous section were established under the a priori assumption that a local CLF with desired properties could be obtained. If the linearization of the considered nonlinear control system is asymptotically null-controllable (this does not hold, e.g., for the null-controllable Brockett nonholonomic integrator, see [72 §10.2] and [12 Example 5.2]), one can use a matrix Riccati equation approach to numerically construct a quadratic CLF and a linear stabilizing feedback for the linearized system, which also serve locally for the original system (see, e.g., [54 Section 3] and [11 Section 1]). Analytical construction of local CLFs may in general be a difficult task, if one first considers the ideal case of unconstrained control inputs and tries to exactly find the corresponding global CLF (by using, e.g., the results of [2] §9.4 [3 Chapter 5]), which can then work locally in case of pointwise control constraints. Besides, for a number of well-known continuous-time mechanical models, the local or global asymptotic stabilization properties of certain feedbacks are derived by means of nonstrict Lyapunov functions, such that the right-hand sides in the related infinitesimal decrease conditions vanish not only at the origin [8–10, 73–75]. However, Definition 2.5 of CLFs and the sufficient conditions for local asymptotic null-controllability used in Remark 2.10 include strictness. Note also that, if one explicitly finds a nonsmooth local CLF satisfying the assumptions of Section 2, and a software implementation of a direct approximation method [43–52] can be launched for the related exit-time optimal open-loop control problems, the nonsmoothness may still cause significant numerical issues.

This section describes the theoretical constructions that were previously introduced in the conference paper [57] and could help to approximate global CLFs for some classes of nonlinear control systems without using any local CLFs. Here we also discuss the proofs omitted in [57] and provide a qualitative comparison with the constructions of Section 2.

Let us consider the control system (1) and indicate the required assumptions.

Since it is not asserted that a suitable local CLF can be obtained, some new conditions on the running cost $g(\cdot, \cdot)$ have to be imposed (they do not appear in Section 2). Furthermore, it is convenient to assume the boundedness of the set of pointwise control constraints. First, Assumption 2.1 is adopted. It is also supposed that a local asymptotic null-controllability property holds in a weak or strong form as follows (see [5 Section 2]).

**Assumption 3.1.** $0_n \in U$, $f(0_n, 0_m) = 0_n$, and one of the following two conditions holds (the second condition strengthens the first one and is called the small control property):

1) there exist positive constants $r, \bar{u}$ and a function $\beta(\cdot, \cdot) \in \mathcal{KL}$ such that $\overline{B}_r(0_n) \subset G$ and, for any $x_0 \in B_r(0_n)$, there is a control strategy $u_{x_0}(\cdot) \in U$ satisfying

$$
\|u_{x_0}(\cdot)\|_{L^\infty(0, +\infty), U} \leq \bar{u},
$$

$$
\|x(t; x_0, u_{x_0}(\cdot))\| \leq \beta(\|x_0\|, t) \quad \forall t \geq 0;
$$

2) there exists a constant $r > 0$ and a function $\beta(\cdot, \cdot) \in \mathcal{KL}$ such that $\overline{B}_r(0_n) \subset G$ and, for any $x_0 \in B_r(0_n)$, there is a control strategy $u_{x_0}(\cdot) \in U$ satisfying

$$
\|x(t; x_0, u_{x_0}(\cdot))\| + \|u_{x_0}(t)\| \leq \beta(\|x_0\|, t) \quad \forall t \geq 0
$$

(this implies (47) with $\bar{u} = \beta(r, 0)$).

**Remark 3.2.** If $0_m \in \text{int} U$, $f(0_n, 0_m) = 0_n$, the function $f(\cdot, \cdot)$ is continuously differentiable, and the linearization

$$
\dot{x}(t) = A x(t) + B u(t), \quad t \geq 0, \quad u(\cdot) \in U,
\quad A \overset{\text{def}}{=} D_u f(0_n, 0_m) \in \mathbb{R}^{n \times n}, \quad B \overset{\text{def}}{=} D_u f(0_n, 0_m) \in \mathbb{R}^{n \times m},
$$

of the system (1) is asymptotically null-controllable, then (1) admits a locally stabilizing linear feedback according to [69 §5.8, Theorem 19], and Item 2 of Assumption 3.1 therefore holds.

**Remark 3.3.** Due to [70 Proposition 7], $\beta(\cdot, \cdot) \in \mathcal{KL}$ implies the existence of two functions $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty$ satisfying

$$
\beta(\rho, t) \leq \alpha_2(\alpha_1(\rho) e^{-t}) \quad \forall \rho \geq 0 \quad \forall t \geq 0.
$$
For example, if $C_9, C_{10}$ are positive constants, $v(\cdot) \in K_{\infty}$, and
\[
\beta(\rho, t) = C_9 v(\rho) e^{-C_{10} t} \quad \forall \rho \geq 0 \quad \forall t \geq 0,
\]
then one can choose
\[
\alpha_1(\rho) = (v(\rho))^{1/C_{10}}, \quad \alpha_2(\rho) = C_9 \rho^{C_{10}} \quad \forall \rho \geq 0 \quad \forall t \geq 0
\]
in order to fulfill the estimate \[39\] in the equality form.

Proposition 3.4. [5 Proposition 2.3] Let Assumptions 2.1 and 3.1 hold. The region of asymptotic null-controllability $D_0$ is an open domain containing the closed ball $B_r(0_n)$. Furthermore, $D_0$ is weakly invariant in the sense that, for any $x_0 \in D_0$, there exists $u_{x_0}(\cdot) \in U$ satisfying $x(t; x_0, u_{x_0}(\cdot)) \in D_0$ for all $t \geq 0$.

Next, let us formulate the conditions on the running cost $g(\cdot, \cdot)$ in an infinite-horizon optimal control problem leading to a global CLF in line with the results of [5] Sections 3 and 4. A typical initial candidate for checking these conditions is $g(x, u) = (\lambda_1/2) \|x\|^2 + (\lambda_2/2) \|u\|^2$ with constants $\lambda_1 > 0$, $\lambda_2 > 0$. One first has to determine appropriate functions $\beta(\cdot, \cdot)$, $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ that characterize the local stabilization asymptotics according to Assumption 3.1 and Remark 3.3.

Assumption 3.5. Let $\alpha^{-1}_1(\cdot)$ be the inverse of the function $\alpha_2(\cdot)$ introduced in Remark 3.3 and take the constants $r, \tilde{u}$ from Assumption 3.1. The following properties hold:

1) $\overline{G \times U} \ni (x, u) \mapsto g(x, u) \in [0, +\infty)$ is a nonnegative continuous function, and, for any $R > 0$, there exists $C_{5,R} > 0$ satisfying the condition (24) (these are Items 1, 2 of Assumption 2.16);

2) for any $R > 0$, one has
\[
\inf \left\{ g(x, u) : (x, u) \in \overline{G \times U}, \|x\| \geq R \right\} > 0;
\]

3) if Item 2 of Assumption 3.1 (the small control property) is not asserted, then there exist positive constants $C_{11}, C_{12}$ such that
\[
g(x, u) \leq C_{11} \left( \alpha^{-1}_2(\|x\|) \right)^{C_{12}} \forall x \in \overline{B}_r(0_n) \forall u \in \overline{B}_u(0_m) \cap U;
\]

4) if Item 2 of Assumption 3.1 holds, then the condition (50) is weakened to
\[
g(x, u) \leq C_{11} \left( \alpha^{-1}_2(\|x\| + \|u\|) \right)^{C_{12}} \forall x \in \overline{B}_r(0_n) \forall u \in \overline{B}_u(0_m) \cap U,
\]

where $C_{11}, C_{12}$ are positive constants;

5) there exists a constant $C_{13} > 0$ satisfying
\[
g(x, u) \geq C_{13} \|f(x, u)\| \quad \forall (x, u) \in \left\{ (x', u') \in \overline{G \times U} : \|x'\| \geq 2r \text{ or } \|u'\| \geq 2\tilde{u} \right\}
\]

(we take $\tilde{u} = \beta(r, 0)$ if Item 2 of Assumption 3.1 holds).

Note the difference between Items 2–5 of Assumption 3.5 on one hand, and Items 3, 4 of Assumption 2.16 together with Assumption 2.19 on the other.

Introduce the infinite-horizon optimal control problem
\[
V_0(x_0) \overset{\text{def}}{=} \inf_{u(\cdot) \in U'} \left\{ \left. \int_0^{+\infty} g(x(t; x_0, u(\cdot)), u(t)) \, dt \right| \in [0, +\infty) \cup \{+\infty\} \forall x_0 \in G. \right\}
\]

Similarly to (40), consider the Kruzhkov transformed value function
\[
u_0(x_0) \overset{\text{def}}{=} 1 - e^{-V_0(x_0)} \in [0, 1] \quad \forall x_0 \in G
\]

with the convention $e^{-(+\infty)} \overset{\text{def}}{=} 0$.

Similarly to [5] Propositions 3.3, 3.5, 3.6 and Remark 4.2, one can obtain the following result that represents the domain of asymptotic null-controllability $D_0$ via the value functions $V_0(\cdot), \nu_0(\cdot)$ and indicates the CLF property for $V_0(\cdot)$ in $D_0$.

Theorem 3.6. Let Assumptions 2.1, 3.1 and Items 1–4 of Assumption 3.5 hold. Then the domain of asymptotic null-controllability is represented as
\[
D_0 = \{ x_0 \in G : V_0(x_0) < +\infty \} = \{ x_0 \in G : \nu_0(x_0) < 1 \}.
\]
If, moreover, Item 5 of Assumption 3.5 holds, then the restriction of \( V_0(\cdot) \) to \( D_0 \) is a CLF for the system \( \{1\} \), and the following statements in particular hold:

- \( V_0(\cdot) \) is continuous on \( D_0 \); \( v_0(\cdot) \) is continuous on \( G \);
- \( \{x_0 \in G : V_0(x_0) = 0\} = \{x_0 \in G : v_0(x_0) = 0\} = \{0\}_n \);
- for any sequence \( \{x^{(k)}\}_{k=1}^\infty \subseteq G \) satisfying either

\[
\lim_{k \to \infty} \text{dist}(x^{(k)}, \partial D_0) = 0 \quad \text{or} \quad \lim_{k \to \infty} \|x^{(k)}\| = +\infty,
\]

one also has

\[
\lim_{k \to \infty} V_0(x^{(k)}) = +\infty \quad \text{and} \quad \lim_{k \to \infty} v_0(x^{(k)}) = 1.
\]

The next theorem can be established similarly to [5, Theorem 4.4] and in fact extends the classical Zubov method for constructing Lyapunov functions [11] to the problem of weak asymptotic null-controllability. Due to the compactness of \( U \), there is no need to adopt [5, Hypothesis (H6)], which states that, for any \( x \in G \) and \( \{u^{(k)}\}_{k=1}^\infty \subseteq U \) satisfying \( \lim_{k \to \infty} \|u^{(k)}\| = +\infty \), one also has

\[
\lim_{k \to \infty} \frac{\|g(x, u^{(k)})\|}{1 + \|f(x, u^{(k)})\|} = +\infty.
\]

**Theorem 3.8.** Under Assumptions 2.1, 3.1 and 3.5, the transformed value function \( v_0(\cdot) \) is the unique bounded viscosity solution of the following boundary value problem for the HJB equation:

\[
\begin{cases}
\max_{u \in U} \{ -\langle Dv_0(x), f(x, u) \rangle - (1 - v_0(x))g(x, u) \} = 0, & x \in G, \\
v_0(0_n) = 0. 
\end{cases}
\]

For numerical purposes, it is helpful to approximate the infinite-horizon optimal control problem [5] by an exit-time problem. If a local CLF and its level sets are not available, the approach of Section 2 cannot be used. The exit-time problem is then stated with respect to the closed ball \( \overline{B}_\delta(0_n) \) with center \( x = 0_n \) and sufficiently small radius \( \delta \in (0, r) \) (see Fig. 2) and recall that the constant \( r \) was introduced in Assumption 3.1:

\[
V_\delta(x_0) \overset{\text{def}}{=} \inf \left\{ \int_0^T g(x(t; x_0, u(\cdot)), u(t)) \, dt : u(\cdot) \in U, \, x(T; x_0, u(\cdot)) \in \overline{B}_\delta(0_n) \text{ at some } T \in [0, +\infty) \right\}
\]

\[
\forall x_0 \in G \quad \forall \delta \in (0, r].
\]

The convention \( \inf \emptyset = +\infty \) is adopted as before. With the help of the notation

\[
T_\delta(x_0, u(\cdot)) \overset{\text{def}}{=} \inf \left\{ T \in [0, +\infty) : x(T; x_0, u(\cdot)) \in \overline{B}_\delta(0_n) \right\} \quad \forall x_0 \in G \quad \forall u(\cdot) \in U
\]

(58)
for the exit times, the value function (57) can also be determined by

\[
V_\delta(x_0) = \inf_{u(\cdot) \in U} \left\{ \int_0^{T_\delta(x_0, u(\cdot))} g(x(t; x_0, u(\cdot)), u(t)) \, dt \right\} \quad \forall x_0 \in G \quad \forall \delta \in (0, r]
\]

(note that the running cost is nonnegative according to Item 1 of Assumption 3.5). Consider also the Kruzhkov transformed function

\[
v_\delta(x_0) \overset{\text{def}}{=} 1 - e^{-V_\delta(x_0)} \in [0, 1] \quad \forall x_0 \in G \quad \forall \delta \in (0, r].
\]

A key result on approximating the infinite-horizon problem (51) by the exit-time problem (57) (or, equivalently, by (59)) can now be derived.

**Theorem 3.9.** Under Assumptions 2.1, 3.1 and 3.5, the following properties hold:

1) the domain of asymptotic null-controllability can be represented as

\[
D_0 = \{ x_0 \in G : V_\delta(x_0) < +\infty \} = \{ x_0 \in G : v_\delta(x_0) < 1 \} \quad \forall \delta \in [0, r]
\]

(according to Definition 2.4 it is obvious that \(D_0\) does not depend on \(\delta\));

2) \(v_\delta(x_0) \to v_0(x_0)\) uniformly on \(G\) as \(\delta \to +0\);

3) \(V_\delta(x_0) \to V_0(x_0)\) uniformly on every compact subset of \(D_0\) as \(\delta \to +0\);

4) for every \(\delta \in (0, r), V_\delta(\cdot)\) is locally Lipschitz continuous in \(D_0\) (and therefore differentiable almost everywhere in \(D_0\) with respect to the Lebesgue measure in \(\mathbb{R}^n\)) if the Petrov condition

\[
\min_{u \in U} \langle x, f(x, u) \rangle < 0 \quad \forall x \in \partial B_\delta(0_n) = \{ x' \in \mathbb{R}^n : \| x' \| = \delta \}
\]

holds.

**Proof.** For any \(\delta \in (0, r],\) the condition \(x_0 \in G \setminus D_0\) yields the absence of state trajectories \(x(\cdot; x_0, u(\cdot))\) corresponding to \(u(\cdot) \in U\) and reaching the target ball \(\overline{B}_\delta(0_n) \subseteq \overline{B}_r(0_n)\) in finite time, while such trajectories exist if \(x_0 \in D_0\) (recall Definition 2.4, Assumption 3.1 and the inclusion \(\overline{B}_r(0_n) \subset D_0\) from Proposition 3.4). This and the property (53) lead to Item 1. Note that Item 3 would follow from Item 2, because

\[
V_\delta(x_0) = -\ln(1 - v_\delta(x_0)) \quad \forall x_0 \in D_0 \quad \forall \delta \in [0, r]
\]

by virtue of the relations (52), (53) and (60). Besides, Item 4 can be established similarly to Item 1 of Proposition 2.20 and the last sentence in Theorem 2.21.

It hence remains to verify Item 2. If \(x_0 \in G \setminus D_0,\) the statement follows directly from Item 1 and the representation (53). Now consider arbitrary \(x_0 \in D_0\) and \(\delta \in (0, r].\) Then the set of control strategies \(u(\cdot) \in U\) satisfying \(T_\delta(x_0, u(\cdot)) < +\infty\) is nonempty. Due to Proposition 3.7 one has

\[
v_0(x_0) = \inf_{u(\cdot) \in U} \left\{ 1 - \mu(x_0, u(\cdot), T_\delta(x_0, u(\cdot))) + \mu(x_0, u(\cdot), T_\delta(x_0, u(\cdot))) \cdot v_0(x(T_\delta(x_0, u(\cdot)); x_0, u(\cdot))) \right\}.
\]
By using the nonnegativity of the running cost, as well as the formulas (55), (59) and (60), one arrives at

\[
0 < \mu(x_0, u(\cdot), T) \leq 1 \quad \forall u(\cdot) \in \mathcal{U} \quad \forall T \geq 0,
\]

\[
v_\delta(x_0) = \inf_{u(\cdot) \in \mathcal{U}} \left\{ 1 - \mu(x_0, u(\cdot), T_\delta(x_0, u(\cdot))) \right\}. \tag{62}
\]

Next, the obtained relations (61), (62) lead to

\[
v_0(x_0) \leq \inf_{u(\cdot) \in \mathcal{U}} \left\{ 1 - \mu(x_0, u(\cdot), T_\delta(x_0, u(\cdot))) \right\} + \sup_{u(\cdot) \in \mathcal{U}} \left\{ \mu(x_0, u(\cdot), T_\delta(x_0, u(\cdot))) \cdot v_0 \left( x \left( T_\delta(x_0, u(\cdot)); x_0, u(\cdot) \right) \right) \right\}
\]

and, consequently,

\[
0 \leq v_0(x_0) - v_\delta(x_0) \leq \sup_{u(\cdot) \in \mathcal{U}} \left\{ \mu(x_0, u(\cdot), T_\delta(x_0, u(\cdot))) \cdot v_0 \left( x \left( T_\delta(x_0, u(\cdot)); x_0, u(\cdot) \right) \right) \right\} \leq \sup_{u(\cdot) \in \mathcal{U}} v_0 \left( x \left( T_\delta(x_0, u(\cdot)); x_0, u(\cdot) \right) \right) \leq \max_{y \in \bar{B}_\delta(0_n)} v_0(y). 
\]

In order to complete the proof, it now suffices to use the property

\[
\lim_{\delta \to +0} \max_{y \in \bar{B}_\delta(0_n)} v_0(y) = 0,
\]

which follows from the equality \(v_0(0_n) = 0\) and the continuity of \(v_0(\cdot)\) on \(G\) mentioned in Theorem 3.6. \(\square\)

**Remark 3.10.** In contrast with the global CLF characterization in Theorem 2.21 involving a local CLF, the approximating value function \(V_\delta(\cdot)\) for a fixed sufficiently small \(\delta \in (0, r]\) leads to the so-called practical stabilization in \(D_0\) (see, e.g., [6, Subsection 2.11]), but not to the asymptotic one. Indeed, the exit-time optimal control problem (57) is stated without using a local CLF and therefore does not allow to obtain stabilizing control actions in the target ball \(\bar{B}_\delta(0_n)\).

In order to ensure the existence of optimal control strategies and to use Pontryagin’s principle for the exit-time problem (57) with \(x_0 \in D_0 \setminus \bar{B}_\delta(0_n)\) and \(\delta \in (0, r]\), we also need Assumptions 2.22 and 2.24. Note that the case when \(x_0 \in \bar{B}_\delta(0_n)\) with \(\delta \in (0, r]\) is trivial and yields \(T_\delta(x_0, u(\cdot)) = 0\) for all \(u(\cdot) \in \mathcal{U}\).

The existence result can be verified similarly to Theorem 2.25.

**Theorem 3.11.** Let Assumptions 2.1, 3.1, 3.5 and 2.22 hold. For any fixed initial state \(x_0 \in D_0 \setminus \bar{B}_\delta(0_n)\) and parameter \(\delta \in (0, r]\), there exists an optimal control strategy for the exit-time problem (57) or, equivalently, for (59).

**Remark 3.12.** Under Assumptions 2.1, 3.1, 3.5, 2.22 and 2.24 Pontryagin’s principle for the exit-time problem (57) with a fixed initial state \(x_0 \in D_0 \setminus \bar{B}_\delta(0_n)\) and a fixed parameter \(\delta \in (0, r]\) can be formulated similarly to Theorem 2.26, but with the difference that now the terminal set appears as the ball \(\bar{B}_\delta(0_n)\) and can be reduced to the sphere \(\partial B_\delta(0_n)\), while the terminal cost vanishes. One should consequently have

\[
T^* = T_\delta(x_0, u^*(\cdot)), \quad \|x^*(T^*)\| = \delta, \quad p^*(T^*) \in N \left( x^*(T^*); \bar{B}_\delta(0_n) \right) = \{ x \in \mathbb{R} : x \geq 0 \}
\]

in the modified characteristic boundary value problem. \(\square\)

The following characteristics based representation is established similarly to Theorem 2.29. For the Hamiltonian and set-valued extremal control map, the notations (56) and (41) are still used.
Theorem 3.13. Let Assumptions 2.1, 3.1, 3.5, 2.22 and 2.24 hold. For any initial state $x_0 \in D_0 \setminus \overline{B}_\delta(0_n)$ and parameter $\delta \in (0, r]$, the Kruzhkov transformed value $v_\delta(x_0)$ defined by (57)–(60) is the minimum of $\begin{aligned} T_{\delta}(x_0, u^*(t)) \end{aligned}$ over the solutions of the characteristic Cauchy problems

\[
\begin{aligned}
\dot{x}^*(t) &= \mathcal{D}_x \mathcal{H}(x^*(t), u^*(t), p^*(t), \bar{p}^*) = f(x^*(t), u^*(t)), \\
\dot{p}^*(t) &= -\mathcal{D}_x \mathcal{H}(x^*(t), u^*(t), p^*(t), \bar{p}^*) \\
&= -\left(D_{x^*} f(x^*(t), u^*(t))\right)^T p^*(t) - \bar{p}^* \mathcal{D}_x g(x^*(t), u^*(t)), \\
\bar{u}^*(t) &\in U^*(x^*(t), p^*(t), \bar{p}^*), \\
t &\in \left\{ [0, T_{\delta}(x_0, u^*(\cdot))], \ T_{\delta}(x_0, u^*(\cdot)) < +\infty, \right. \\
&\left. [0, +\infty), \ T_{\delta}(x_0, u^*(\cdot)) = +\infty, \right\}
\end{aligned}
\]

for all extended initial costates

\[
(p_0, \bar{p}^*) \in \{(p, \bar{p}) : p \in \mathbb{R}^n, \ \bar{p} \in \{0, 1\}\}.
\]

Moreover, the same value is obtained when minimizing over the bounded set

\[
(p_0, \bar{p}^*) \in \{(p, \bar{p}) \in \mathbb{R}^n \times \mathbb{R} : \|(p, \bar{p})\| = 1, \ \bar{p} \geq 0\},
\]

or even over its subset

\[
(p_0, \bar{p}^*) \in \{(p, \bar{p}) \in \mathbb{R}^n \times \mathbb{R} : \|(p, \bar{p})\| = 1, \ \bar{p} \geq 0, \ \mathcal{H}(x_0, p, \bar{p}) = 0\}.
\]

4 | CONCLUSION

Based on the theoretical results of this paper, one can use direct or characteristics based numerical frameworks to approximate CLFs and associated feedbacks independently at different initial states. As was discussed in the introduction, this enables for attenuating the curse of dimensionality and selecting arbitrary bounded regions and grids in the state space. Parallel computations can also be arranged. Furthermore, the stabilizing control action at any isolated state can be directly retrieved either as the initial value of an approximate optimal open-loop control strategy computed via a direct method, or by the corresponding representation in Pontryagin’s principle (recall (41)) with the initial state and an approximate optimal initial costate. The latter can be obtained via a characteristics based method or as an appropriate costate estimate building on direct collocation [51][52]. Possibly unstable approximations of CLF gradients are therefore not needed.

As was also noted in the introduction, even if the curse of dimensionality is attenuated, the curse of complexity is still a formidable issue when constructing global or semi-global solution approximations in high-dimensional regions. Regarding a real-time MPC implementation, if a current state lies outside the sublevel set $\Omega_c$ of the local CLF $V_{\text{loc}}(\cdot)$ or outside the target ball $\overline{B}_\delta(0_n)$ for practical stabilization (recall Remark 3.10), the time to compute the related control action by solving a direct or characteristics based nonlinear programming problem is not negligible, so that the MPC strategy may not be properly updated in real time if the resulting delay is unacceptably large.

In order to make online computation of CLF values, costates, and control actions faster though less accurate, one can employ a sparse grid framework such that a sparse grid along with relevant data on it is generated beforehand in offline mode and a certain interpolation algorithm using this data is then executed online [36]. Note that Smolyak type sparse grids built in [36] have a special hierarchical configuration with predetermined nodes. Scattered grids and related interpolation methods [76][78] are more flexible in the sense that new nodes can be directly included in such a grid to improve the approximation (machine learning may help to select suitable new nodes). Other data-driven techniques for fast real-time control evaluation rely on neural networks and polynomial regression [79][81]. It is also reasonable to involve gradient (costate) enhanced interpolation of CLFs, which may enable acceptable accuracy with a significantly smaller number of grid nodes compared to the case when the gradient information is not incorporated [76][77]. Another important issue is to ensure appropriate values of a CLF approximation and
its gradient on the whole terminal set of the considered exit-time optimal control problem. The second part of our work proposes a numerical approach that addresses the mentioned issues. This approach combines gradient enhanced modifications of the Kriging and inverse distance weighting frameworks for scattered grid interpolation and allows for convenient offline incorporation of new data with the aim to improve constructed interpolants.

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APPENDIX

Proof of Lemma 2.13

Let \( x \in E \) and \( \zeta \in D^*_p \varphi(x) \). According to the Lipschitz continuity of \( \varphi(\cdot) \) and the definition of a proximal subgradient (see [60] p. 5), there exist positive numbers \( \epsilon_0, \sigma \) (depending on \( x \)) such that \( B_{\epsilon_0}(x) \subseteq E \) and

\[
C \|x' - x\| \geq \varphi(x') - \varphi(x) \geq \langle \zeta, x' - x \rangle - \sigma \|x' - x\|^2
\]

for all \( x' \in B_{\epsilon_0}(x) \). For every \( i \in \{1, 2, \ldots, n\} \), let \( e_i \in \mathbb{R}^n \) be such that its \( i \)-th coordinate equals 1 and all the other coordinates vanish. Take arbitrary \( i \in \{1, 2, \ldots, n\} \) and \( \epsilon \in (0, \epsilon_0) \). Then (A.1) reduces to \( C \geq \langle \zeta, e_i \rangle - \sigma \epsilon \) for \( x' = x + \epsilon e_i \) and to \( C \geq -\langle \zeta, e_i \rangle - \sigma \epsilon \) for \( x' = x - \epsilon e_i \). As \( \epsilon \to +0 \), one obtains \( |\langle \zeta, e_i \rangle| \leq C \). This leads directly to (13). \( \square \)

Proof of Theorem 2.25

Consider the optimal control problem (23) or (35) with a fixed initial state \( x_0 \in D_0 \setminus \Omega_c \). In line with Proposition 2.18, one has \( V(x_0) < +\infty \). Fix an arbitrary \( \epsilon > 0 \). By \( U^*_\epsilon(x_0) \), denote the set of all \( u(\cdot) \in U^* \) for which the cost is not greater than \( V(x_0) + \epsilon \). The control subclass \( U^*_\epsilon(x_0) \) obviously contains a minimizing sequence. Recall also the notation (22). By the definition of \( U^*_\epsilon(x_0) \), one has \( T_{\Omega}(x_0, u(\cdot)) < +\infty \) for all \( u(\cdot) \in U^*_\epsilon(x_0) \). If one proves that the integral funnel

\[
\{ (t, x(t; x_0, u(\cdot))) : t \in [0, T_{\Omega}(x_0, u(\cdot))], u(\cdot) \in U^*_\epsilon(x_0) \}
\]

is contained in some compact set \( K \subseteq \mathbb{R}^{n+1} \), then including the constraint that admissible integral trajectories should lie in \( K \) will not change the infimum in the considered optimal control problem, while this will allow for using the general existence theorem of [41] §9.3. Thus, it remains to establish the boundedness of (A.2). According to Remark 2.2, it suffices to verify that the set of exit times \( \{ T_{\Omega}(x_0, u(\cdot)) : u(\cdot) \in U^*_\epsilon(x_0) \} \) is bounded. Due to the definition of \( U^*_\epsilon(x_0) \) and Item 4 of Assumption 2.16, any \( u(\cdot) \in U^*_\epsilon(x_0) \) satisfies

\[
C_6 T_{\Omega}(x_0, u(\cdot)) + c \leq \int_0^{T_{\Omega}(x_0, u(\cdot))} g(x(t; x_0, u(\cdot)), u(t)) \, dt + c \leq V(x_0) + \epsilon
\]

with a constant \( C_6 > 0 \), which leads to the estimate

\[
T_{\Omega}(x_0, u(\cdot)) \leq \frac{V(x_0) + \epsilon - c}{C_6}
\]

and therefore completes the proof. \( \square \)

Proof of Proposition 2.30

By using the representation of directional derivatives of minimum functions (see, e.g., [82] Theorem I.3.4], which considers maximum functions, but can be similarly reformulated for minimum functions), one can verify that

\[
\frac{d}{dt} H(x^*(t), p^*(t), \tilde{p}^*) = 0 \quad \text{for almost all} \quad t \in I(x_0, u^*(\cdot)).
\]
Since $x^*(\cdot)$ and $p^*(\cdot)$ are absolutely continuous on every compact subset of $I(x_0, u^*(\cdot))$ and $H(\cdot, \cdot, \cdot)$ is Lipschitz continuous on every compact subset of $G \times \mathbb{R}^n \times \mathbb{R}$ (due to, e.g., [82, Remark I.3.2]), the function $I(x_0, u^*(\cdot)) \ni t \mapsto H(x^*(t), p^*(t), \tilde{p})$ is also absolutely continuous on any compact subset of $I(x_0, u^*(\cdot))$. Hence, (A.3) implies that the latter function is constant on $I(x_0, u^*(\cdot))$. \qed
\[ V_{\text{loc}}(x) < c \text{ in } \text{int } \Omega_c, \quad V_{\text{loc}}(x) > c \text{ outside } \Omega_c, \]
\[ V(x) = V_{\text{loc}}(x) = c \text{ on } l_c \]

\[ x(T_{\Omega_c}(x_0, u(\cdot)); x_0, u(\cdot)) \in l_c \]

\[ x_0 \in \mathcal{D}_0 \]
$x(T_\delta(x_0, u(\cdot)); x_0, u(\cdot))$

$x_0 \in \mathcal{D}_0$

$0_n$

$\delta$

$V_\delta(x)$
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Yegorov, I; Dower, PM; Gruene, L

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