Fast Extremum Seeking for Online Calibration of Engines with Variable Natural Gas Composition

Jalil Sharafi

Submitted in total fulfilment of the requirements of the degree of Doctor of Philosophy

Department of Mechanical Engineering
THE UNIVERSITY OF MELBOURNE

April 2016
Abstract

There is a growing interest worldwide in using alternative fuels, motivated by the potential benefits they can offer regarding emissions and energy security. Among the alternative fuels, natural gas has received special attention due to its vast proven reserves across the world. On the other hand, the composition variation of natural gas poses a challenge for the conventional lookup table-based engine control technology.

Recently, Extremum Seeking (ES) theory has been considered by the automotive research community as a solution to cope with the composition variation. Although the existing experimental results have demonstrated improved efficiency by employing ES in steady state operation, yet none of them displays neither fast convergence nor the ability to handle transient engine operation. This thesis addresses these gaps by proposing two extensions to ES theory, namely the fast model-based ES and the multiplexed ES scheme, and experimentally demonstrates the performance improvement achieved in calibration of a CNG-fueled engine.

Conventional ES implementations rely on a type of time-scale separation tuning which leads to slow optimization relative to the plant dynamics. While this approach treats the plant as a black box by making no assumption on the plant dynamics, the slow optimization is not desirable in some applications. Recently, two different solutions are proposed to address this issue: the model-based ES approach that uses a parametrized model of the plant to benefit from warm starting the estimation as well as using a large class of optimization algorithms; and the fast ES approach that makes use of partial plant knowledge to accelerate estimation. As the first contribution, this thesis develops a fast model-based ES scheme to combine the advantages of a model-based approach and the accelerated performance of fast ES. Experimental results show improved fuel economy by reducing
the convergence time of spark timing calibration by an order of magnitude compared with a conventional model-based ES algorithm. Similar results are demonstrated in injection duration calibration using an extension of fast ES developed in this thesis.

While the existing ES implementations only consider steady state optimization, in some applications such as the motivating application of this thesis a time-varying extremum must be tracked. This is of particular interest in online calibration of engines over a typical driving cycle. As the second contribution of this thesis, a novel multiplexed ES algorithm is introduced to track the time-varying extremum caused by a measurable disturbance. The experimental results show the successful calibration of spark timing for a CNG-fueled engine over the New European Driving Cycle (NEDC).
Declaration

This is to certify that

1. the thesis comprises only my original work towards the PhD,

2. due acknowledgement has been made in the text to all other material used,

3. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Jalil Sharafi, April 2016
First and foremost, I would like to express my sincere gratitude to my supervisors Professor Chris Manzie and Dr. Will Moase. In particular, I would like to thank Chris for his patience, motivation, and immense knowledge and his continuous support of my Ph.D study, and Will whose deep knowledge of the field was both intimidating and inspirational. Also, I would like to thank Professor Michael Brear for his valuable support throughout my experiments.

I am grateful to those who helped over the course of my PhD, in particular: Dr. Peter Dennis who commissioned the test rig and also trained me in the operation of the engine dynamometer; Timothy Broomhead who was my lab partner and helped tremendously with my experiments; Dr. Alireza Mohammadi who was a true “yes man” and was never too busy to help; Simon Hager and Geert van Kollenburg who spent their internships with me and assisted with my experiments.

I thank my fellow officemates for providing a pleasant environment: Vincent Bachtiar who never said no to my coffee invites, Ronny Kutadinata with whom I had great stimulating discussions, Kuan Waey Lee, Dr. Rohan Shekhar, Brett Bishop, Chih Feng Lee, Kashyapa Sirinandara, Changfu Zou, Kaixiang Wang, Frank Liu, Yaqi Zhu, Gokul Sankar Sastry.

Last but not least, I am thankful for my loving parents for their encouragement throughout my studies. I am also deeply grateful to my wife for her endless love, continuous support and her companionship throughout our PhD journeys.
Contents

1 Introduction .......................................................... 3
  1.1 Background ......................................................... 3
  1.2 Thesis Layout ...................................................... 5

2 Literature Review .................................................. 7
  2.1 Natural Gas as an Alternative Fuel ......................... 7
  2.2 Natural Gas Composition Variation and Engine Performance 8
  2.3 Engine Calibration Techniques ............................... 10
  2.4 Extremum Seeking Control ..................................... 13
    2.4.1 Black-Box Extremum Seeking ......................... 15
    2.4.2 Grey-Box Extremum Seeking ......................... 19
    2.4.3 Fast Black-Box Extremum Seeking ................... 22
    2.4.4 Extremum Seeking for Plants with Exogenous Disturbance 26
  2.5 Extremum Seeking in Engine Control ...................... 27
  2.6 Summary ......................................................... 28
  2.7 Research Aims .................................................... 29

I Theory .............................................................. 33

3 Fast Model-Based Extremum Seeking on Hammerstein Plants 35
  3.1 System Description .............................................. 35
    3.1.1 Plant ......................................................... 36
    3.1.2 Filter ......................................................... 36
    3.1.3 Optimizer ................................................... 37
    3.1.4 Parameter Estimator ...................................... 38
  3.2 Stability Analysis .............................................. 39
  3.3 Simulation Example ............................................ 45
    3.3.1 Gradient Descent Optimizer .......................... 46
    3.3.2 Model-based Optimizer ................................. 47
  3.4 Conclusion ....................................................... 49

4 Fast Extremum Seeking on a Class of Hammerstein-Wiener Plants 51
  4.1 Fast Extremum Seeking Controller Development ............ 51
  4.2 Simulation ....................................................... 54
4.3 Conclusion .................................................. 56

5 Multiplexed Extremum Seeking for Systems with Exogenous Inputs 57
5.1 System Description ......................................... 58
  5.1.1 Plant .................................................. 58
5.2 Stability Analysis .......................................... 59
  5.2.1 Preliminary Analysis for a Perturbed Gradient System .... 60
  5.2.2 Analysis for a Static Map Extremum Seeking ............... 61
  5.2.3 Analysis for a Dynamical Plant ......................... 64
5.3 Multiplexed Extremum Seeking Scheme ....................... 66
  5.3.1 Fast Multiplexed Extremum Seeking Scheme ............... 72
5.4 Simulation Example ........................................ 74
5.5 Conclusion .................................................. 77

II Implementation to Online Calibration of a CNG-fueled Engine 79

6 Experimental Setup .......................................... 81
  6.1 Test Fuels ................................................ 83
  6.2 Test Conditions .......................................... 84
    6.2.1 Steady State Operation .............................. 84
    6.2.2 Transient Operation .................................. 85

7 Fast Extremum Seeking for Optimization of Brake Specific Fuel Consumption 87
  7.1 Spark Timing Optimization ................................ 87
    7.1.1 Identification of Engine with Brake Torque as Output and Spark Timing as Input ............. 88
    7.1.2 Controller Overview .................................. 90
    7.1.3 Experimental Results ................................ 93
  7.2 Injection Duration ........................................ 95
    7.2.1 Identification of Engine with Brake Torque as Output and Injection Duration as Input ........ 95
    7.2.2 Controller Overview .................................. 98
    7.2.3 Experimental Results ................................ 98
  7.3 Conclusion ................................................ 101

8 Multiplexed Extremum Seeking for Calibration of Spark Timing in a CNG-fueled Engine 103
  8.1 Experimental Results ..................................... 104
    8.1.1 Controller Development ............................ 105
    8.1.2 Constant Torque-Speed Test ......................... 108
    8.1.3 Driving Cycle Results ............................... 110
  8.2 Conclusion ................................................ 112
9 Contributions and Future Work

9.1 Contributions to Extremum Seeking Theory ........................................ 115
  9.1.1 Fast Model-based Extremum Seeking on Hammerstein Plants ........ 115
  9.1.2 Fast Black-box Extremum Seeking on a Class of Hammerstein–Wiener
       Plants .................................................................................. 115
  9.1.3 Analysis of Black-box ES in the Presence of Disturbance .......... 116
  9.1.4 Multiplexed ES for Systems with Time-varying Extremum Caused
       by a Measurable Disturbance ................................................. 116

9.2 Contributions to Online Calibration of CNG-fueled Engines .......... 117
  9.2.1 Experimental Demonstration of Fast Model-based ES in Set-point
       Calibration of Spark Timing ....................................................... 117
  9.2.2 Experimental Demonstration of Fast ES in Set-point Calibration of
       Injection Duration ................................................................. 117
  9.2.3 Experimental Demonstration of Multiplexed ES in Optimization of
       Spark Timing over a Driving Cycle .......................................... 118

9.3 Future Work .................................................................................. 119
  9.3.1 Extension to Consider Input Constraints in fast ES ................. 119
  9.3.2 Further Extension of Fast Model-based ES to Use the Whole Range
       of Dither Frequencies ............................................................. 119
  9.3.3 Further Extension of Fast Model-based ES to Multi-input Single-
       output plants .................................................................. 119
  9.3.4 Using In-cylinder Pressure Transducer for Feedback ............. 120
  9.3.5 Considering Emission Constraints in Set-point Calibration ....... 120

A 121
  A.1 Proof of Theorem 3.1 - Stability .................................................. 121
     A.1.1 Unperturbed-interconnected System .................................. 122
     A.1.2 Perturbation Effect ......................................................... 126
  A.2 Proof of Theorem 3.1 - Convergence Rate .................................. 128
  A.3 Lyapunov Function for Isolated Estimator ................................. 129

B 131
  B.1 Averaging Analysis for Static Map Extremum Seeking ............... 131
  B.2 Proof of Theorem 5.1 .................................................................. 132
  B.3 Sketch of Proof Of Theorem 5.2 .................................................. 140
  B.4 Proof of Theorem 5.3 .................................................................. 140
List of Figures

1.1 Oil price from 1947 to 2011 [25]. .................................................. 4

2.1 The conventional engine control technique [37]. ............................ 10
2.2 Cylinder pressure versus crank angle for overadvanced spark timing of 50 CAD BTDC, MBT spark timing of 30 CAD BTDC, and retarded spark timing of 10 CAD BTDC [43]. ................................. 11
2.3 Steady-state mapping between the spark timing and brake torque for pure Methane and Gas B at engine operating point of 800 rpm and 30 Nm [84]. .......................... 11
2.4 The effect lambda (defined as the fraction of the actual air-fuel ratio to the stoichiometric air-fuel ratio) on natural gas engine performance [15]. ............................... 12
2.5 Plant with input-output steady state map exhibiting an extremum. ........ 13
2.6 Perturbation based extremum seeking controller [90]. ...................... 14
2.7 Black-box ES scheme[11]. .......................................................... 15
2.8 ES feedback loop which illustrates tuning strategy proposed in [66]. Dither amplitude \( a \), dither frequency \( \Omega \) and \( \delta \) are (small positive) tuning parameters. \( K \), \( \omega_L \) and \( \omega_H \) are positive \( O(1) \) constants. .......................................................... 17
2.9 Grey-Box ES Controller. \( k \) and \( \Omega \) have to be sufficiently small to split the time-scale between system dynamic, parameter estimator and optimizer algorithm. \( D_N(\cdot) \) denotes a vector of derivatives required by the optimization algorithm. .......................... 21
2.10 Time-scale separation in ES is shown, where the estimator is estimating \( Q'(u) \), or the unknown parameter \( \theta \) of \( Q(u, \theta) \) in a model-based ES. .............................. 23
2.11 Time-scale separation in a “fast” black-box ES. ............................ 23
2.12 Extremum Seeking With Compensator. Design parameters are dither amplitude \( a \) and frequency \( \Omega \), compensator \( C(s) \) and phase lag in demodulation signal \( \phi \). ......................... 24
2.13 The fast ES scheme [80]. ............................................................ 24
3.1 The proposed ES algorithm. ....................................................... 36
3.2 Static input-output map, \( h(u) \). .................................................. 45
3.3 Persistency of excitation level \( \alpha_0 \) versus dither amplitude \( a \) in logarithmic scale (circle). The slope of the fitted line is 12 which is equal to \( 2n_\theta \). The calculation is done for \( \bar{\bar{u}} = -1, 0, 1 \) rad. .......................................................... 46
3.4 Optimizer output \( \bar{u} \) versus \( \Omega t \) (rad) for \( \bar{\bar{u}}(0) = 0.7 \) and \( \bar{\bar{u}}(0) = -1.75 \) (rad) at \( \Omega = 40 \) (rad/s) (solid), and \( \Omega = 200 \) (rad/s) (dashed). .......................... 46
3.5 Optimizer output \( \bar{u} \) versus \( \Omega t \) (rad) for \( \bar{\bar{u}}(0) = 0.7 \) and \( \bar{\bar{u}}(0) = -1.75 \) (rad) at \( \Omega = 40 \) (rad/s) (solid), and \( \Omega = 200 \) (rad/s) (dashed). ......................... 49
3.6 Optimizer output \( \bar{u} \) versus \( \Omega t \) (rad) for \( \bar{\bar{u}}(0) = 0.7 \) and \( \bar{\bar{u}}(0) = -1.75 \) (rad) at \( \Omega = 40 \) (rad/s) (solid), and \( \Omega = 200 \) (rad/s) (dashed). .......................... 49
4.1 ES controller structure .................................................. 52
4.2 ES output $\bar{\eta}$ (ms/cycle) versus $\Omega t$ (rad) for $\bar{\eta}_0 = 6.6$ and $\bar{\eta}_0 = 5$ (s/cycle) at $\Omega = 10$ (Hz) (solid), $\Omega = 100$ (Hz) (dashed), and $\Omega = 1000$ (Hz) (dash-dotted). ............................................... 54
4.3 The bode plot of the output dynamics $F_e(s)$ and a delay function of 0.112 seconds. ........... 56
4.4 The offset in ES from the optimal value for different compensation. .......................... 56
5.1 ES scheme influenced by an exogenous input signal vector $w(t)$. .......................... 59
5.2 $u^*(w)$ when $w(t) \in \mathbb{R}$. ................................................. 66
5.3 Multiplexed ES scheme. .................................................. 67
5.4 The activation function $\sigma_t(w(t))$. The hysteresis regions are coloured grey. ........ 68
5.5 ES (solid), ES$_2$ (dashed), and ES$_3$ (dash-dotted) activations are shown for a specific $w(t)$. 68
5.6 The activation strategy is shown for $\Gamma \subset \mathbb{R}^2$. The hysteresis regions are colored grey. ES$_1$ (solid), ES$_2$ (dashed), and ES$_3$ (dash-dotted) activations are shown for an arbitrary $w(t)$. . 69
5.7 The schematic of multiplexed ES scheme implemented with fast ES schemes. ......... 73
5.8 A triangular wave with the frequency of 0.5 Hz and a random sequence with the sampling time 2 seconds are used as the exogenous input $w(t)$. ......................... 75
5.9 $u_1(t)$ versus $t_1$ when $w(t)$ is a triangular wave with the frequency of 0.5 Hz (dashed line) and 5 Hz (solid line). The region between the dash-dotted lines highlights the locus of $u_1^*(\bar{w})$ for $\bar{w} \in [2, 4]$. .................. 75
5.10 Plant state $x$ versus $t_1$ when $w(t)$ is a triangular wave with the frequency of 0.5 Hz (grey line) and 5 Hz (black line). .................................................. 75
5.11 The ES output $\bar{\eta}(t)$ shown for a triangular exogenous input of frequency 0.5 Hz (dashed line) and 5 Hz (solid line) ............................. 75
5.12 The ES output $\bar{\eta}(t)$ shown for a triangular exogenous input of frequency 0.5 Hz for multi-plexed ES with $M = 3$ (dashed line) and $M = 1$ (grey line) against $u^*(w(t))$ (black solid line). .................................................. 76
6.1 Transient dynamometer test facility .................................. 82
6.2 The layout of the hardware configuration. ...................................... 83
6.3 Flow curve shows the linear performance of the injectors for injection durations more than 2.5 ms, and constant fuel line pressure of 3.2 bar. The manufacturer’s specification is given for a fuel line pressure of 3 bar. .................................................. 84
6.4 New European Driving Cycle and the corresponding engine speed and torque for the test engine. Shaded regions correspond to idle operations ............................... 85
7.1 Steady-state mapping between Brake torque (Nm) and spark timing (CAD BTDC) for the test gas at WWM operating point. Smaller step size was chosen for points closer to the MBT. 89
7.2 The structure of engine as a plant as perceived by the controller. ...................... 90
7.3 The torque response of the engine for two step changes in spark timing. The raw data (grey line), the filtered output of the torque sensor (blue line), and the approximate Hammerstein model output (black line) are shown. The dashed line indicates the moment of changing the spark timing. .................................................. 90
7.4 The Experimentally acquired bode plot versus that of a 4th order Butterworth filter with a cutoff frequency of 2.5 Hz. The slight mismatch in the amplitude is due to the interference from other inputs, such as cam movements, during the experiments. ...................... 91
7.5 Controller schematic. .................................................. 93
7.6 Model-based ES performance for a dither frequency of 0.4 Hz, dither amplitude of 1.5 CAD BTDC, and optimizer gain of 0.25. ................................. 94
7.7 Fast model-based ES performance for a dither frequency of 1.7 Hz, dither amplitude of 1.5 CAD BTDC, and optimiser gain of 0.25. ......................... 94
7.8 Fast model-based ES performance for a dither frequency of 1.9 Hz, dither amplitude of 1.5 CAD BTDC, and optimizer gain of 0.25. ............................. 95
7.9 Steady-state mapping between BSFC and injection durations for two gas compositions at the operating point of 1500 rpm and 6.25% throttle position. For the test gas, the optimum injection duration occurs around 5.8 (ms/cycle), whereas it is around 7 (ms/cycle) for Gas B specified in Table 6.2. ............................................................... 96
7.10 Engine as a plant with the injection duration as input and BSFC as output. ......................... 96
7.11 Experimentally acquired Bode plot versus the identified system dynamics. ......................... 98
7.12 The observer-based ES controller structure to optimize volume-based BSFC. .................. 99
7.13 Black-box ES [111] performance for the dither frequency of 0.25 Hz and optimizer gain of 0.4. 100
7.14 Fast ES scheme performance for the dither frequency of 1 Hz and optimizer gain of 0.4. .... 100
7.15 Fast ES scheme performance for the dither frequency of 1 Hz and optimizer gain of 1. Increasing the optimizer gain led to an undesirable performance. ........ 100
7.16 Fast ES scheme performance for the dither frequency of 2 Hz and optimizer gain of 0.4. Increasing the dither frequency led to an undesirable performance. ..... 100
8.1 The conventional Table Lookup (TLU) approach (a), versus the proposed adaptive approach (b). ................................................................. 104
8.2 MBT spark timing for the test gas. The operating point of 1380 rpm and 0.29 load is marked with a star. ................................................................. 107
8.3 The subregions defined for multiplexed ES. ................................................................. 108
8.4 The implemented activation function. The figure is not drawn to scale. .......................... 108
8.5 Spark timing ES at a constant torque-speed mode. ...................................................... 109
8.6 The distribution of the operating points in speed-load plane during a New European Driving Cycle (NEDC) sampled at 1 second intervals. The black lines show the grids, and the grey lines define the hysteresis regions. Also the indexes are shown by numbers in each region. ................................................................. 110
8.7 Activation signals are shown for the subregions during one NEDC and four extra UDC. 111
8.8 The output of the ESs are plotted against their respective time axes t. The dashed line shows the interpolated MBT value from the lookup table for the test gas, and the black line marks the end of the first NEDC round. .......................... 113
Mathematical Preliminaries

It is convenient to review some preliminaries that will be used throughout the analyses in the following chapters.

- The continuous function $\alpha : [0, a) \rightarrow \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{K}$ if it is nondecreasing and $\alpha(0) = 0$.

- $\alpha(\cdot)$ is class $\mathcal{K}_\infty$ if it is class $\mathcal{K}$ and $\lim_{r \rightarrow \infty} \alpha(r) = \infty$.

- A function $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{L}$ if it is decreasing and $\lim_{t \rightarrow \infty} \sigma(t) = 0$.

- A function $\beta : [0, a) \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{KL}$ if it is class $\mathcal{K}$ with respect to the first argument and class $\mathcal{L}$ with respect to its second argument.

- The identity matrix of size $n$ is denoted by $I_n$.

- Denote the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms by $\| \cdot \|$ and $\| \cdot \|_\infty$ respectively.

- A ball in $\mathbb{R}^n$ is defined as $B^n(r) = \{x \in \mathbb{R}^n : \|x\| \leq r\}$.

- A quantity, $x \in \mathbb{R}^n$, is $O(c)$ if there exist $(c^*, K) \in \mathbb{R}_{\geq 0}^2$ such that $\|x\| \leq Kc$ for all $c \in (0, c^*)$.

- Also in this chapter $[X; Y]$ is used instead of $[X^T Y^T]^T$.

- Where $F(s)$ denotes an arbitrary transfer function, $F(s)[u(t)]$ represents its time-domain response to input $u(t)$.

- The gradient of a continuous function $f(x, y)$ with respect to $x$ is denoted as $f_x'$. 
Chapter 1
Introduction

1.1 Background

In the recent years, concerns are growing worldwide over the use of petroleum-based fuels as the main source of energy. Currently more than 97% of all the energy consumed in the transport sector in Australia remains petroleum-based [91]. It contributes to approximately 17% of national CO$_2$ equivalent emissions [20]. Other emissions such as NOx, soot and particulate matter are also issues which pose health threats, particularly in dense urban areas. Apart from transportation, diesel generators are commonly used to supply electricity when connections to the grid are not feasible and to supply peak-shaving and back-up electricity. Moreover, oil is a non-renewable resource and the limited reserves are concentrated in politically unstable countries which has implications for national energy security. Figure 1 shows that from 1990-2007 the relative cost of oil has risen nearly 300% [25]. Considering these reasons, alternative fuels are desirable to reduce our dependency on petroleum-based fuels.

Among the alternative fuels known today, Natural Gas (NG) has received special attention, as vast reserves can be found all around the world. It can be used easily as a fuel in current SI engine vehicles by modifying their fuel supply and adjusting the injection system. Methane (CH$_4$) is the main constituent of NG (typically between 85 – 95%) and the balance consists of heavier hydrocarbons and inert gases such as carbon dioxide (CO$_2$) and nitrogen (N$_2$). Methane, has the highest Hydrogen/Carbon ratio among other hydrocarbon fuels. Thus, using NG may decrease CO2 emission. The downside of using NG as a fuel is that, in unrefined forms its composition can vary markedly [16, 62, 73].
Natural gas composition variation can change its combustion properties, which subsequently affects the engine performance.

In traditional engine control strategies, engine input set-points are tuned in a lengthy calibration process. In the calibration process, steady-state engine performance and emission maps are obtained versus engine inputs for different operating conditions. These maps are then used to select the input set-points that result in an optimal performance while meeting the requirements of engine safety, drivability and emission regulations. Subsequently, the optimum set-points are stored at the Engine Control Unit (ECU) in a lookup table format. Since the calibration procedure is carried out for a specific fuel composition, therefore this method may fail to give the optimal performance if the fuel composition varies from the one used during the calibration.

Fuel composition affects combustion properties such as stoichiometric air-fuel ratio, energy content, burn rate, and knock resistance. Therefore, the optimal engine inputs can vary as a result of the variation in the fuel composition [55,77]. Subsequently it can give rise to a suboptimal performance if the engine is operated on a different composition than the one used for engine calibration.

Fuel composition variation is hard and expensive to measure and also adds more
complexity to the engine control and calibration process [2, 94]. Hence, there is a clear need to implement some form of online optimization to maximize the benefits promised by alternative fuels. Extremum Seeking (ES) theory can be utilized as a good candidate to achieve this goal.

One limitation in the current implementations of ES has been its slow convergence speed. In several of recent experimental studies, ES has been applied to optimize various engine inputs with respect to fuel efficiency in steady-state calibration scenarios [41, 68, 96]. These results indicated that by improving convergence speed significantly, there is a clear benefit in fuel economy, particularly during the transient operation. More importantly, in automotive applications the engine is very rarely run in steady state for long periods of time, thus there is motivation to not only have fast convergence, but also extend the methodology to handle fast varying operating point.

1.2 Thesis Layout

In the next chapter, the literature concerning the alternative fuels and the problems associated with their composition variability will be reviewed. Then the engine calibration techniques will be discussed focusing on the effect of fuel composition variation in this context. The rest of the chapter reviews the available extremum seeking schemes available in the literature with regards to the tuning guidelines leading to the time-scale separation requirement.

The following content is presented in two main parts. In the first part that includes Chapters 3–5 the theoretical results are presented that provide novel ES techniques for a large class of applications; while the second part that includes Chapters 6–8 reports the experimental results demonstrating the performance of the developed ES schemes in the calibration of a CNG-fueled engine in different scenarios.

In Chapter 3, a new fast model-based ES is developed to achieve high estimation rate by employing a high dither frequency in a model-based framework. It is assumed that the plant has a Hammerstein structure, which approximates many engineering applications. A substantial portion of this chapter has been published in [102] and [106].
In Chapter 4, the fast black-box approach is extended to incorporate a special class of Hammerstein-Wiener plants. Later, in Chapter 7 it is shown that this class of systems can approximate the engine input-output behavior considering the injection duration as input and the Brake Specific Fuel Consumption (BSFC) as output. A substantial portion of this chapter has been published in [101].

Chapter 5 presents two theoretical contributions. First, the effect of a slowly varying disturbance on a widely used black-box extremum seeking introduced in [111] is investigated. The main result of the first part can be considered as an extension of the black-box ES to the case where the plant is subject to exogenous disturbances. Second, a new ES scheme is developed that can be used to estimate the extremum input across the range of a measurable exogenous disturbance. The proposed algorithm can be regarded as an adaptive feed-forward controller. A substantial portion of this chapter has been submitted for publication in [103] and [105].

In Chapter 6 the experimental setup is described. Afterwards, in Chapter 7, the fast ES schemes developed in Chapter 3 and Chapter 4 are experimentally tested for steady-state set-point calibration of a CNG-fueled engine with unknown fuel composition. First, the fast model-based technique of Chapter 3 is implemented to calibrate spark timing. Then, the fast black-box technique of Chapter 4 is tested to calibrate injection duration.

In Chapter 8, the multiplexed ES scheme developed in Chapter 5 is experimentally tested to calibrate spark timing of the CNG engine over the New European Driving Cycle. A substantial portion of this chapter has been submitted for publication in [104].

Chapter 9 reviews the major contributions of this thesis and discusses future investigations that would further supplement the presented results.
Chapter 2

Literature Review

2.1 Natural Gas as an Alternative Fuel

As mentioned in the previous chapter, there is a growing interest worldwide in using alternative fuels to replace petroleum motivated by the potential benefits they can offer regarding emissions and energy security. Given the limited known oil reserves and increasing demand, the forecasted petroleum price increase has been another major driver behind this policy [95]. Among alternative fuels, natural gas is currently the second most consumed alternative fuel in transportation sector (after liquified petroleum gas) with approximately 17 million vehicle running on Compressed Natural Gas (CNG) worldwide [55,64]. One of the reason for the adoption of natural gas is its vast proven reserves across the world, which are larger than crude oil [95].

Natural gas can be used in vehicles either in a bi-fuel strategy with gasoline or diesel, or by employing dedicated mono-fuel CNG engines. For conventional gasoline-fueled spark-ignition vehicles to run on natural gas only the fueling system needs to be modified. Currently, after-market retrofitted CNG vehicles constitute the majority of the CNG vehicles. Studies have shown between 10 to 20% reduction in the produced power of retrofitted CNG vehicles [6,7]. The power loss in a bi-fuel vehicle is mostly attributed to the lack of proper engine control technology [62] and the lower volumetric efficiency, reported as up to 10% [97,100]. To address the latter problem, direct injection [118], turbo charging [58] and super charging [8] techniques have been shown to improve output power of natural gas engines. Also, dedicated mono-fuel CNG engine may achieve comparable brake power and efficiency [107] by employing higher compression ratios [97].
Regarding the emissions, natural gas is known to produce significantly less particulate matters than diesel in compression-ignition engines [31]. Moreover, methane as the main constituent of natural gas has the highest hydrogen to carbon ratio, therefore it can potentially produce less CO$_2$ emission than diesel and gasoline [6,7]. On the other hand, the literature shows a lack of consensus about the emission characteristics of CNG vehicles, particularly on the produced NO$_X$ level. Some research has shown reduced tailpipe NO$_X$ level at stoichiometric combustion using Three Way Catalytic (TWC) converter [7, 47]. On the other hand, both higher compression ratios and lean-burn combustion technology can result in higher NO$_X$ compared with gasoline. For a complete survey on emission characteristics of natural gas engines refer to [17, 63] and the references therein. Although emission considerations have an important role in the adoption of natural gas, they will not be discussed any further in this study.

2.2 Natural Gas Composition Variation and Engine Performance

Natural gas does not refer to a specific gas or fuel characteristics. The primary constituent of natural gas is methane that in volume percentage can vary between 85 to 98 percent, while the other constituents may include heavier hydrocarbons such as ethane and propane, and inert gases such as nitrogen and carbon dioxide. In fact, the natural gas composition depends mainly on the source of extraction, and varies across different geographical locations [1]. Table 2.1 shows the variation in the composition of natural gas extracted from different production areas. Furthermore, the local distribution companies occasionally modify the natural gas composition for reasons such as peak shaving [19,71]. For example, propane (or propane-air mixture) may be added to keep the gas properties within the acceptable limits during peak demand periods [87]. Since currently most of the natural gas used for vehicles is supplied from the pipeline gas, this may influence the performance of CNG-fueled vehicles.

The natural fuel composition affects its properties such as density, stoichiometric air-fuel ratio, energy content [62,77]. It also affect the combustion characteristics of the gas such as burning rate and knock resistance [12, 27, 87, 99, 116]. For example, the addi-
2.2 Natural Gas Composition Variation and Engine Performance

Table 2.1: Specifications of natural gas in different locations. Gas 1 is extracted from Indonesia [77], Gas 2 and Gas 3 are pipeline natural gas in Victoria and Queensland states in Australia, Gas 4 is extracted from Abu Quir field in Egypt [27], Gas 5 is from Bidboland field in Iran [55]

<table>
<thead>
<tr>
<th>Component</th>
<th>Formula</th>
<th>Volume fraction (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Gas 1</td>
</tr>
<tr>
<td>Methane</td>
<td>CH$_4$</td>
<td>90.87</td>
</tr>
<tr>
<td>Ethane</td>
<td>C$_2$H$_6$</td>
<td>5.81</td>
</tr>
<tr>
<td>Propane</td>
<td>C$_3$H$_8$</td>
<td>2.38</td>
</tr>
<tr>
<td>Butane</td>
<td>C$<em>4$H$</em>{10}$</td>
<td>0.89</td>
</tr>
<tr>
<td>Heavy hydrocarbons</td>
<td>C$<em>n$H$</em>{2n+2}$ ($n \geq 5$)</td>
<td>0.03</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>N$_2$</td>
<td>0.02</td>
</tr>
<tr>
<td>Carbon dioxide</td>
<td>CO$_2$</td>
<td>0</td>
</tr>
</tbody>
</table>

The effects of composition variation on engine performance has been evident since the early studies on the adoption of natural gas for road vehicles in 1980s. In [53], the authors demonstrated the penalties of using different natural gas compositions when the engine was optimized for a specific fuel composition. A complete survey on the early studies prior to 2000 can be found in [72]. There is still an ongoing concern over the effect of using different natural gas compositions and the resulting variability in the performance and emissions of the road vehicles [56, 73]. The composition variation poses a challenge for the conventional engine control techniques, which will be discussed next. Before moving on, it is noted that other alternative fuels such as LPG [98], ethanol-gasoline blends [39], and biogas [47] also suffer from composition variability.

...
2.3 Engine Calibration Techniques

Conventional engine control relies mainly upon static feedforward components that determine set-point values for various engine inputs [37]. As shown in Figure 2.1, the feedforward blocks contain static calibration maps that determine engine inputs set-points based on the engine speed and engine load\(^1\), generally referred to as engine operating point. They guarantee the fast adjustment of engine inputs during the highly dynamic operation. For some engine parameters, the inaccuracies in the static maps are compensated by the complementary feedback loops using sensor measurements.

The calibration maps are obtained in a lengthy steady-state optimization procedure. The objective function is assigned based on the often competing objectives such as fuel consumption, produced emissions, and drivability. The optimization is started by dividing the engine speed-load plane in a grid-based fashion [96]. At each grid knot, the engine is then mapped by manually changing the engine inputs and recording the performance measures at steady-state, after the engine transient dynamics have settled. Eventually, the unique set of optimal engine input values are selected and stored in the calibration maps, also referred to as Table-Lookup Unit (TLU). When the engine is operated between the grid points, the stored values are interpolated using bilinear interpolation [50].

The spark timing as one of the important engine parameters influences the torque output (hence the efficiency) of the combustion in a 4-stroke spark ignition engine. A typical spark timing calibration involves locating the ignition moment that produces the Maximum Brake Torque (MBT). The existence of MBT spark timing results from the existence of a delay time between the ignition event and the moment of maximum pressure

\(^1\)Engine load is defined as the fraction of air inducted into the engine relative to its theoretical maximum value.
inside the cylinder [43]. Therefore, to achieve the MBT, ignition must occur in the compression stroke before the Top Dead Center (BTDC) so that the maximum pressure takes place early after the top dead center in the power stroke. Figure 2.2 illustrates that an earlier (overadvanced) ignition will result in more pumping loss, while a later (retarded) ignition will result in reduction in the maximum pressure. As shown in Figure 2.3, a 10 degree error in MBT spark timing results in around 5\% reduction in the brake torque. As such, the spark event must be timed properly to achieve the MBT.

Since, the burn rate changes with the fuel composition for natural gas [12, 27, 87, 116], it is expected that the MBT spark timing changes with the fuel composition. This fact is demonstrated for two different natural gas composition in Figure 2.3 [84].

Air-fuel ratio is another engine operating parameter affecting the efficiency of the combustion as well as the produced emissions. For gasoline engines equipped with TWC converters, it is kept at stoichiometric to ensure high catalytic efficiency [37]. However, natural gas has a higher flammability range, which can extend the combustion to relative air-fuel ratios around 1.6 [26]. As shown in Figure 2.4, operating the engine at lean-burn condition can lower engine-out emissions (particularly for NO\(_X\)) while achieving better
efficiency, both resulting from lower combustion temperature [15,57]. Just as stoichiometric combustion, in lean-burn combustion it is extremely important to accurately control the air-fuel ratio [18,57]. In lean-burn operation slight inaccuracies in air-fuel ratio control can lead to losing efficiency, misfire, and higher emissions [32]. More importantly, finding the optimal set-point\(^2\) (which is a trade-off between the fuel consumption and emissions) is challenging, as it varies with the fuel composition [100].

Accurate air-fuel ratio and spark timing control is necessary to guarantee the optimal performance of a CNG engine [32,62,97]. However, because the set-point calibration is done for one specific fuel composition, therefore it delivers suboptimal performance when a different composition is used. This is a major challenge for the adoption of most alternative fuels with variable composition. For some alternative fuels such as gasoline-ethanol blends there are existing methods to detect the mixture composition [2,94]. However, even in that case adding a new variable to the composition brings more complexity to the calibration procedure and engine control, and thus is undesirable.

The composition variation issue can be mitigated if the engine controller is equipped with a means of adapting to the unknown fuel composition. The variability in optimal inputs motivates the use of adaptive methods to adjust the engine inputs to unknown fuel compositions. Due to the highly complex nature of combustion, non-model-based approaches are preferred to deal with the composition variation. In that regards, using Extremum Seeking (ES) methods that will be introduced in the next section is considered

\(^2\)Engine input values stored in lookup tables are referred to as engine set-points.
Figure 2.5: Plant with input-output steady state map exhibiting an extremum.

as a promising approach [41].

### 2.4 Extremum Seeking Control

An optimization problem, in the simplest case, is concerned with finding the extremizing input of a known function. Depending on the complexity of the function, this can be done either analytically or through iterative numerical procedures. Some of these numerical methods, like gradient descent, Newton’s method, and line search use the function curvature information (i.e. its first order derivative or higher) to locate a local extremum iteratively [30]. For example, in a gradient descent algorithm, the current estimate of the local maximum in each step is updated by moving in the opposite direction of the gradient. These methods largely rely upon the availability of the model and can be done off-line.

In some applications the map to be optimized is the steady state input–output characteristic of a dynamical plant, while the underlying dynamics are not completely known. This is depicted in Figure 2.5. Suppose that for each constant input $u$, after a transient the output settles at a constant $y$. Thus, the steady-state map may be expressed as $y = Q(u)$, which is the input-output equilibrium map.

**Assumption 2.1.** The input–output equilibrium map, $y = Q(u)$, has a unique minimum\(^3\) at $u = u^*$.

Now, consider the objective of regulating the input $u$ close to $u^*$. The control techniques used to achieve this goal is called Extremum Seeking (ES). Since a reliable model

\(^3\text{Minimization of } Q(.) \text{ is considered without loss of generality. Maximization of } \hat{Q}(.) \text{ can be done by applying minimization to } Q(.) = -\hat{Q}(.)\).
Literature Review

Figure 2.6: Perturbation based extremum seeking controller [90].

does not exist a priori, ES algorithms use the input-output measurements in a feedback structure to shift the plant output progressively towards the extremum. One such method is shown in Figure 2.6. In this widely used implementation, local curvature information of the function is estimated by observing the perturbation signal effect, which is then used by an optimization algorithm to update the input.

The invention of ES dates back to 1922, when Leblanc proposed a new control mechanism to improve efficiency of a resonant circuit in an overhead transmission line by adjusting the variable inductance [109]. According to Sternby [108], ES was a popular research topic as a branch of adaptive control in 1950s and 1960s. In that period what is considered here as an ES loop was called with different names, such as extremum control system, optimizing control system, extremum seeking regulator and hill-climbing system [38]. Among the various applications of ES in that period of time were combustion process, solar cell and radio telescope antenna adjustment (see the references in [108]). Despite the early advances in the theory of ES and several reported application, a rigorous mathematical stability analysis as well as general guidelines for tuning the parameters were lacking. In 2000, Krstić and Wang [66] provided the first rigorous stability analysis for an ES scheme. Interests in ES revived afterwards and many successful application of ES have emerged (see [109] for a comprehensive review of these).

The level of knowledge about the plant defines different classes of ES. In the so called black-box approach, the plant model is assumed to be unknown [5]; whereas in the model-based (grey-box) approach the plant model is known but some parameters in the model are unknown [89]. In the sequel, relevant literature in ES will be reviewed. In the first subsection black-box ES and the relevant stability results are discussed. It is followed by the introduction of the model-based ES design method and the results published so
far. The third subsection presents the attempts that have been made to accelerate convergence speed in ES. The final subsection is devoted to discuss the available results that consider ES applied on plants subject to an exogenous disturbance.

### 2.4.1 Black-Box Extremum Seeking

As mentioned before, in black-box ES the plant dynamics and its steady-state behavior are assumed to be completely unknown. Therefore the following assumption is central to all black-box ES schemes:

**Assumption 2.2.** The input–output equilibrium map, \( Q(u) \), is unknown. However the input, \( u \), and the output, \( y \), are available for measurement.

One of the most popular black-box approaches discussed in the ES literature is based on perturbing the input with a small amplitude signal, also known as the dither signal. The perturbed output demodulated by the dither signal can be shown to have an indication of the local gradient. This scheme is shown in Figure 2.7, which is taken from [11]. For the sake of simplicity in explanation, the plant dynamics are neglected so that the input–output map can be resented as a static map \( y = Q(u) \).

![Figure 2.7: Black-box ES scheme][11]

It is assumed that the static map satisfies Assumption 2.1. \( \bar{u} \) denotes the estimate of \( u^* \). If \( k \) is sufficiently small, \( \bar{u} \) varies slowly relative to the perturbation signal. Then the output can be expanded using Taylor series expansion

\[
y(u) \approx Q(\bar{u}) + aQ'(\bar{u}) \sin(t) + O(a^2) \quad (2.1)
\]
If the dither signal amplitude is sufficiently small, the $O(a^2)$ term can be neglected. Passing through High Pass Filter (HPF) would remove slowly varying $Q(\bar{u})$. The remaining signal is demodulated using the perturbation signal and is low-pass filtered. The cutoff frequency of both HPF and LPF are assumed to be smaller than $\Omega$. Thus, $\dot{\bar{u}}$ would be approximately equal to:

$$\dot{\bar{u}} \approx -\frac{ka^2 Q'(\bar{u})}{2}$$

(2.2)

The above equation approximates gradient descent for $\bar{u}$ and after a finite time it will converge to a vicinity of $u^*$. In real world applications, the static map to be optimized is often the steady state input-output map of a dynamic plant. In the early applications of ES, because the optimum input was assumed to vary slowly, system dynamics were often neglected and the input-output relation was considered merely as a static function $y = Q(u)$. Thus, many of the developed schemes had been derived from numerical optimization, where only the measurement of the cost function in the output and/or its gradient are required [108].

In 2000, Krstić and Wang [66] proved that the local stability of the ES scheme shown in Figure 2.8, can be guaranteed through the careful selection of the tuning parameters in the loop. The scheme is adopted from [11]. The dynamical plant is represented as:

$$\dot{x} = f(x, u),$$

(2.3a)

$$y = h(x),$$

(2.3b)

where $u$ represents either the system input or a controller parameter taken as an input by the ES controller. The functions $f(\cdot, \cdot)$ and $h(\cdot)$ are unknown, therefore the input-output equilibrium map, $y = Q(u)$, holds in Assumption 2.2. Moreover, the following assumption on the plant input–output dynamics (2.3) is considered,

**Assumption 2.3.** The input-output equilibrium map is locally exponentially stable.

Krstić and Wang showed that, if the initial condition is close to this extremum point, for a sufficiently small value of $a$, $\Omega$ and $\delta$, the integrator output (which acts as a gradient descent optimizer) will eventually converge to an $O(a + \Omega + \delta)$ neighborhood of $u^*$. When the equilibrium map is globally asymptotically stable for all $u \in \mathbb{R}$, Tan et al. [111] have
taken the same tuning strategy and proved a non-local version of the results in [66]. To explain the idea of the results in [111], it is worthwhile to define Semi-globally Practically Asymptotic (SPA) stability prior to the discussion of the non-local result.

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x)
\end{align*}
\]

**Figure 2.8:** ES feedback loop which illustrates tuning strategy proposed in [66]. Dither amplitude \(a\), dither frequency \(\Omega\) and \(\delta\) are (small positive) tuning parameters. \(K, \omega_L\) and \(\omega_H\) are positive \(O(1)\) constants.

**Definition 2.1.** A system \(\dot{x} = f(t, x, \epsilon)\) is SPA stable uniformly in \(\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_j), j \in \{1, ..., l\}\) if there exist a KL function \(\beta\) such that the following holds. For each pair of strictly positive numbers \((\Delta, \nu)\), there exist real numbers \(\epsilon_k^* = \epsilon_k^*(\Delta, \nu)\), \(k = 1, 2, ..., j\) and for each fixed \(\epsilon_k \in (0, \epsilon_k^*)\), \(k = 1, 2, ..., j\) there exist \(\epsilon_i = \epsilon_i(\epsilon_1, \epsilon_2, ..., \epsilon_{i-1}, \Delta, \nu)\), with \(i = j + 1, j + 2, ..., l\), such that the solutions of the system with the so constructed parameters satisfy

\[
|x(t)| \leq \beta(|x_0|, (\epsilon_1, \epsilon_2, ..., \epsilon_l)(t - t_0)) + \nu,
\]

for all \(t \geq t_0 \geq 0, x(t_0) = x_0\) with \(|x_0| \leq \Delta\).

This definition relates the three main characteristics of an ES controller, namely, convergence rate, accuracy, and Domain Of Attraction (DOA). In terms of SPA, given that the steady state input–output map is globally asymptotically stable, the feedback loop is SPA uniformly in \(\epsilon = (a^2 \Omega \delta)^T\). While the main theorem in [66] shows the relation between the convergence rate and the accuracy of the ES scheme, [111] can explain the relationship between all three performance characteristics and the way they are affected by tuning parameters utilizing SPA stability of Definition 2.1.

\footnote{If the set of real numbers is denoted by \(\mathbb{R}\), then, the continuous function \(\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}\) is of class KL if it is nondecreasing in its first argument and converging to zero in its second argument.}
The specific way of picking tuning parameters in [66, 111] requires the closed-loop to occupy three separate time-scales, with the plant dynamics being the fastest, the estimator dynamics being slower, and the optimizer dynamics being the slowest. More explicitly, by choosing the dither frequency to be very small, the feedback loop components will act slowly relative to the plant dynamics, i.e. the plant appears as a static map.

The choice of the dither signal and how it can affect the performance in an ES feedback loop has been a subject for several studies. In [112] authors studied three different periodic zero-mean perturbations with the same frequency and amplitude to be used for a static map ES case. For this purpose, they investigated sin wave, triangle wave and square wave and it has been established that among those, square signal can give faster convergence rate while other performance characteristics are the same. The choice of square wave dither signal is preferred as it is especially easy to generate in some applications. In [112] authors also investigated the dependency of the accuracy on the excitation signal. The results therein emphasized that in general error in optimization can not approach zero by merely reducing dither amplitude; in fact dither frequency also affects the steady state error. Other researchers have exploited different choices of the perturbation signal. In [74] Manzie and Krstić utilized a stochastic signal to perturb the system, in that way the correlation between the input and output signal gives the required gradient estimation.

As mentioned earlier the ES feedback loops studied in [65, 111] can be interpreted as a gradient estimator in cascade with a gradient descent optimization algorithm. This interpretation helps to think of combining other methods of gradient estimation and continuous optimization algorithms to implement ES. Banaszuk et al. [10] proposed using an observer-based gradient estimator with a gradient descent optimiser for a combustion instability application. The observer-based gradient estimation technique is used in conjunction with a Newton optimization method in [81]. The Newton method optimization algorithm takes into account the second order derivative of the optimising function, therefore the optimizer update rate can be adapted based on the curvature of the function. In general a higher order optimizer can be used given that an appropriate derivative estimator is available. Nesić et al. [90] proposed an $N$th order derivative estimator to be
used in conjunction with an appropriate $N$th order optimizer to achieve ES. They gave the condition under which SPA stability of the ES feedback loop can be achieved.

Using perturbation signal to seek the extremum (perturb and observe) is not the only way of achieving ES. Non-perturbation based approaches were also investigated to implement ES. For example, the sliding mode ES scheme is used mainly in applications where switching behavior is inherent in the problem ([93]). Such applications include anti-lock braking system and maximum power point tracking for photovoltaics [22, 70]. There were other types of non-perturbation based ES scheme in the early studies in 1960s, for example see the self-driving and switching ES algorithms in [11, 108]. They are not considered here, as the lack of robustness prevented them from later developments.

The main advantage of black-box ES approaches is that, no $a$ priori knowledge of the plant is needed except to satisfy Assumptions 2.1 and 2.3. However, reviewing the literature in this area reveals that the black-box ES implementations have an intrinsic limitation of slow convergence. It can not be overcome unless stronger assumptions are considered. The focus of the review over the next subsections is on two methods to address this issue.

### 2.4.2 Grey-Box Extremum Seeking

In black-box ES, little information about the system is needed beforehand to design the ES feedback loop and also little information is collected about the system afterwards. This mainly corresponds to Assumption 2.2. Another class of ES systems that have been referenced in early publications assumes more knowledge of $Q(\cdot)$ [11, 108]. It is assumed that the input-output equilibrium map is a known function of some unknown parameters, denoted as

$$ y = Q(u, \theta) $$

where $\theta \in \Omega_\theta \subset \mathbb{R}^{N_\theta}$ is a fixed unknown parameter vector and $u \in \mathbb{R}^m$ is the input. In this case, Assumption 2.1 still holds but Assumption 2.2 is loosened;

**Assumption 2.4.** The input-output equilibrium map, $Q(u, \theta)$, is known but the parameter vector, $\theta$, is unknown. Moreover, for the given $\theta$ there exists a strict minimum.
For example, in a static map optimization case, the static map can be \( y = \theta_1 Q_1(u) + \theta_2 Q_2(u) \), where \( \theta_i \)'s are unknown parameters to be identified and \( Q_i \)'s are known functions of the input \( u \). The optimizer output is then calculated based on the current estimation of the model \( \hat{y} = \hat{\theta}_1 Q_1(u) + \hat{\theta}_2 Q_2(u) \) obtained by some kind of online parameter estimation. Therefore identifiability (i.e. convergence of \( \hat{\theta}_i \)'s to \( \theta_i \)'s) is an important aspect of these schemes. To ensure identifiability of the parameters, it may be necessary to superimpose a perturbation on the control signal in these schemes.

Assuming structure for the steady-state input-output map allows the use of a wide range of model-based optimization techniques leading to a faster convergence speed; for example using higher order curvature information can be used easily within this framework to improve convergence rate. The essential assumption of the existing knowledge about the system is fulfilled in many applications either by doing open loop tests or physics-based modeling. While the traditional approach to ES is called Black-box (due to its blindness with respect to internal plant dynamics), the later approach is often referred to as “model-based ES” [108] or “ES for systems with parametric uncertainties” [36, 89].

Sternby [108] mentioned “model-oriented” ES as one of four branches of ES and referenced some early theoretical developments by that time. One of the early applications of this method reported in the literature was optimizing Brake Specific Torque (BST) with respect to the spark ignition angle in an internal combustion engine [117]. Based on the past experiences it is known that the BST can be modeled as a quadratic function of spark angle, \( u \), at each speed-load operating point:

\[
BST = \theta_2 u^2 + \theta_1 u + \theta_0, \tag{2.6}
\]

where \( \theta_i \)'s are functions of engine load-speed condition and are subject to change throughout the life cycle of an engine.

Guay et al. [36] considered a different class of ES in which the system states are assumed to be measurable and both system dynamics and objective function are known functions of uncertain parameters \( \theta \in \mathbb{R}^n \). This assumption is particularly stronger than what is required by Assumption 2.4. The objective of the ES is therefore to optimise (artificial) performance measure \( y = Q(x_p, \theta) \) with respect to \( x_p \) subject to the system
2.4 Extremum Seeking Control

Parameter Estimator

\( \dot{\theta} = \Omega \mathcal{G}(\hat{\theta}, u, y) \)

Optimisation Algorithm

\( \bar{u} = k \Omega \mathcal{F}(\mathcal{D}_N(\hat{\theta}, \bar{u}), \bar{u}) \)

Figure 2.9: Grey-Box ES Controller. \( k \) and \( \Omega \) have to be sufficiently small to split the time-scale between system dynamic, parameter estimator and optimizer algorithm. \( \mathcal{D}_N(\cdot) \) denotes a vector of derivatives required by the optimization algorithm.

input-affine dynamics

\[
\begin{align*}
\dot{x}_p &= f_p(x) + F_p(x)\theta + G(x)u, \\
\dot{x}_q &= f_q(x).
\end{align*}
\] (2.7a, 2.7b)

The essential assumption is that the objective function \( y = Q(x_p, \theta) \) is strictly convex for all \( \theta \in \Omega_n \subset \mathbb{R}^{N_\theta} \). There is no assumption on the stability of the plant in this method as \( u \) can be assigned to stabilize the system.

A fairly general framework for design of this type of ES scheme was recently given in [89]. This scheme is shown in Figure (2.9). The plant dynamics can be described by:

\[
\begin{align*}
\dot{x} &= f(\theta, x, u), \\
y &= h(\theta, x).
\end{align*}
\] (2.8a, 2.8b)

The assumption here is that the system dynamics have a known uniformly locally asymptotically stable equilibrium map \( x = l(\theta, u) \). Therefore the input-output map at the steady state would be a function of unknown parameters \( \theta \):

\[
y = h \circ l(\theta, u)
\] (2.9)

Note that, while in the black-box approach the nonlinear function is assumed completely unknown, in this approach \( h \circ l(\cdot) \) is regarded as a known function. This is a restrictive assumption, but allows the use of many model-based optimizers. The main result in [89]
indicates that if each subsystem is stable, then by choosing \( k \) and \( \Omega \) sufficiently small, one can always achieve SPA stability, i.e. for a given \((\Delta, \nu)\) there exist sufficiently small \((k, \Omega)\) such that the state trajectories will satisfy

\[
||[\tilde{x}(t), \tilde{\theta}(t), \tilde{u}(t)]|| \leq \beta(||[\tilde{x}(0), \tilde{\theta}(0), \tilde{u}(0)]||, k\Omega(t - t_0)) + \nu
\]  

(2.10)

when \(|[\tilde{x}(t_0), \tilde{\theta}(t_0), \tilde{u}(t_0)]|| \leq \Delta\). Typically, a dither signal is utilized to ensure identifiability of the unknown parameter. The main result in [89] provides the conditions under which one can combine any parameter estimator algorithm with any model-based optimization scheme to achieve ES.

The model-based approach, firstly, enables a broad range of optimizers to be used in ES, and secondly, allows “warm starting” the parameter estimates with reasonable values based on some partially known properties of the function to improve the convergence time. However, this method requires time-scale separation of the plant dynamics, parameter estimator and optimizer in a similar manner to traditional black-box approaches, and consequently this tuning strategy generally exhibits slow convergence.

### 2.4.3 Fast Black-Box Extremum Seeking

The stability results discussed so far dictate a particular way of adjusting feedback loop parameters. This tuning strategy leads to a time-scale separation between the optimizer, the estimator (gradient estimator or parameter estimator) and the plant dynamics to ensure stability. The resulting time-scale separation is illustrated in Figure 2.10. In this way the optimizer dynamics would be slowest compared to the other dynamics present in the loop. This is an inherent limitation in the implementations of ES [65, 89, 111]. On the other hand, it is clearly desirable to have a fast convergence rate, as the objective function is always related to some real costs and profits. In the context of engine calibration, more fuel is saved if the ES is performed faster. In addition, in some applications the optimized input is time varying, and it is crucial for ES controllers to track the varying extremum.

An alternative approach to address the slow convergence of ES is to avoid the three time-scale tuning of the closed-loop altogether. More explicitly, time-scale separation of
the plant and ES scheme is traditionally achieved by specifying the ES scheme to evolve in the $\Omega t$ time-scale, where $\Omega$ is a small dither frequency. If the requirement for a small dither frequency is relaxed, the estimator can evolve in a faster time-scale than the plant, while the time-scale separation between the optimizer and estimator is retained. This concept is shown in Figure 2.11.

To be able to use a high frequency dither signal, and potentially speed up ES convergence rate, some knowledge of the plant dynamics is needed. Existing results for fast ES typically treat the plant as having a Wiener-Hammerstein (WH) structure as shown in Figure 2.12. This allows any nonlinearity in the plant input–output behavior to be represented within a static mapping which is “sandwiched” between Linear Time Invariant (LTI) input and output dynamics. Krstić [65] gave a local stability result for black-box ES on WH plants with an unknown quadratic nonlinearity and known input and output dynamics. When $F_i(s)$ and $F_o(s)$ are known, ES can be achieved by properly designing the compensator $C(s)$ and tuning the phase lag $\phi$ for any given high dither frequency range. The main result was expanded to a general case with time-varying $u^*$ in [5].

Moase and Manzie [80] later showed that it is possible to achieve arbitrarily fast semi-global convergence for black-box ES on Hammerstein plants with fairly arbitrary static nonlinearities and requiring little more knowledge than the relative degree of the plant output dynamics. Unlike the analysis in [65], the proposed scheme by Moase and Manzie does not require the exact knowledge of the plant dynamics $F_o(s)$. This is formally stated in the following assumptions (while Assumption 2.1 and 2.2 still hold):

Assumption 2.5. The plant can be represented as a nonlinear map, $Q(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ followed by a
stable LTI filter, $F_o(s)$, which has a minimal state space representation

\[
\begin{align*}
\dot{x}_F &= A_F x_F + B_F Q(u), \\
y &= C_F x_F + D_F Q(u),
\end{align*}
\]

(2.11a)

(2.11b)

where $x_F \in \mathbb{R}^{N_F}$. Without loss of generality $F_o(0) = 1$.

In addition, Assumptions 2.1 is modified in [80] with the following assumption.

**Assumption 2.6.** There exist $(u^*, r, K) \in \mathbb{R}^3$ such that

- $Q'(u^* + \sigma)$ is continuous and bounded $\forall \sigma \in [-r, r]$;
- $Q'(u^* + \sigma)/\sigma \geq K$ for all $\sigma \in [-r, r] - \{0\}$.

The fast ES scheme is illustrated in Figure 2.13, which can be expressed as,

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} = \begin{bmatrix}
\Omega (A - LC' - k\Omega C'(\Omega t - \phi)) & \Omega L \\
-k\Omega C'(\Omega t - \phi) & 0
\end{bmatrix} \begin{bmatrix}
\bar{x} \\
y
\end{bmatrix},
\]

(2.12)

where,

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 1 & 0
\end{bmatrix}, \quad C'(\cdot) = \begin{bmatrix}
0 & \sin(\cdot) & \cos(\cdot)
\end{bmatrix}.
\]
In (2.12) $\varphi_F = \arg F_o(i\Omega)$, $a$ and $\Omega$ are the perturbation signal amplitude and frequency respectively, and the observer states can be shown to be,

$$\hat{x} = \begin{bmatrix} \hat{Q}_0 \\ |F_o(i\Omega)|a\hat{Q}'(\bar{u})(\sin(\Omega t + \varphi_F)) \\ |F_o(i\Omega)|a\hat{Q}'(\bar{u})(\cos(\Omega t + \varphi_F)) \end{bmatrix},$$

In the observer states, $\hat{Q}_0$ and $\hat{Q}'$ respectively represent the estimates of the mean-value and the gradient of the map to be optimized. The phase shift $\phi(\Omega)$ is added in demodulation signal vector $C'(\cdot)$ to effectively compensate for the phase shift incurred by the output dynamics $F_o(s)$. The following assumptions are placed on the ES controller.

**Assumption 2.7.** $L$ is chosen so that $A - LC$ is Hurwitz.

**Assumption 2.8.** There exist $J \in \mathbb{R}_{>0}$ and $D \subset \mathbb{R}_{>0}$ such that for all $\Omega \in D$,

$$\text{Re}\{e^{i\phi(\Omega)}F_o(i\Omega)F_1(i\Omega)G(i)\} \geq J. \quad (2.13)$$

**Remark 2.1.** The fast ES approach theoretically allows for $\Omega$ to be selected in any set $D \subset \mathbb{R}_{>0}$ as long as $\phi(\Omega)$ can be chosen to ensure Assumption 2.8 is satisfied. Unlike most traditional ES approaches, time-scale separation between the estimator and the plant dynamics is not necessary.

The inequality (2.13) is satisfied through appropriate design of filters applied to the plant’s output [80]. It only requires the relative degree of the plant output dynamics and the phase shift effect of the $F_o(i\Omega)$ to be known within $\pm \pi/2$ rad. The term “fast ES” was coined by the authors to refer to this class of ES schemes. The idea was further generalized to a Wiener-Hammerstein plant model for high dither frequency [79].

In summary, to achieve a fast convergence rate in ES, it is necessary to have some knowledge about the plant dynamics. This requires more knowledge about the plant than is required by either black-box ES or grey-box ES. Wiener-Hammerstein model used by Krstić [65] and Moase and Manzie. [79] has shown to be a candidate in this regard.

**Remark 2.2.** It is noted that similar tuning strategy (using high dither frequency $\Omega$) is used by another class of ES applied on plants with input affine structure [24]. For input-affine plants, an
ES loop can be developed that requires the dither frequency signal to be sufficiently high in order to drive the closed-loop trajectories close to their Lie bracket approximation. The input-affine requirement is restrictive in some applications.

2.4.4 Extremum Seeking for Plants with Exogenous Disturbance

In the majority of the ES algorithms discussed so far, the underlying assumption is that the plant has a stationary input-output behavior that exhibits an extremum point. However, this assumption does not hold in some applications where the plant is subject to some exogenous input disturbances, \(w\). In essence, this exogenous input may require the steady state extremum in the typical extremum seeking formulations, \(u^*\), to be replaced with an extremum mapping, \(u^*(w)\). Consequently, the extremum value can be quite variable. This scenario is relevant in online recalibration of engines with alternative-fuel in the presence of engine speed and load variation [41], as well as many other energy systems such as: MPPT for a wind turbine in the presence of wind speed variation [33]; MPPT for a fuel-cell in the presence of load variation [14,119].

The inability of traditional ES structure to handle these type of problems has motivated recent developments. A novel approach is proposed in [40] that employs a moving average filter to find the gradient information required by a gradient-based ES. It is used in tuning of a variable gain controller for a 6-DOF motion platform in [48]. The result in [40] exploits the fact that the exogenous input and its effect are periodic which limits its applicability for systems that violate this condition. Similarly, for systems that admit a periodic steady state output, novel ES techniques are proposed in [13,35].

An adaptive feedforward approach is demonstrated in [75] for static plants when the exogenous input disturbance signal is measurable. In this scheme, the map between the exogenous input and the extremum point is continuously identified and incorporated in the feedforward block. This approach does not require the exogenous input to be periodic. However, the stability conditions, particularly the “persistency of excitation” condition needed to ensure identifiability of the map, are overlooked.
2.5 Extremum Seeking in Engine Control

The first application of a ES controller in an engine control setting was reported by Draper and Li in 1955 [23]. They applied what they termed as a “peak holding” method to maximize brake torque of a single-cylinder CFR engine with respect to spark timing and air-fuel ratio, while the fuel flow rate and speed were held constant. In the peak holding method, the ES output was generated proportional to the difference between the current output and the maximum indicated output, which was provided as a input.

Several factors, including the production tolerances and engine wear over time, may cause the static calibration-based engine control to deliver suboptimal performance in a gasoline engine. Wellstead and Scotson, used a model-based ES technique to adapt the MBT spark timing in a spark ignition engine [117]. Brake torque was considered to be a quadratic function of spark timing, the exact shape of which can change in-service and also from part to part. Therefore, a recursive least square estimator was utilized to identify the map, and a Newton method was used to adjust the spark timing to maximize the brake torque. Although a successful demonstration of the approach was given in [117], the tuning strategy and stability issues were not addressed. Similar approach was later used by Larsson and Andersson in [68] to maximize brake torque using spark timing of each cylinders individually, with the relevant stability proofs presented in [69].

Motivated by the increasing number of inputs in modern engines, ES has been considered to facilitate the the complexity of the calibration process. In [96], brake specific fuel consumption was optimized with respect to the spark timing and intake and exhaust cam timings. ES techniques based on Simultaneous Perturbation Stochastic Approximation (SPSA) and the Persistently Exciting Finite Differences (PEFD) algorithm were used, the stability of which was supported by the results presented in [113]. The experimental results have shown successful application of ES, taking about 15 minutes to converge for each fixed operating point.

More recently, the widespread use of alternative fuels has sparked interest in the potential of using ES for online calibration to account for the composition variation. In [41] ES was considered for tuning MBT spark timing in a flex-fuel engine during constant operating condition as well as a very mild sinusoidal transient operation. A model-based
ES approach was used in [83] for online calibration of spark timing for a CNG-fueled engine with an unknown fuel composition. The ES performance was investigated at different constant operating conditions, representing stationary power plant applications. Although successful demonstration of ES was reported in [83] and [41], yet none of them displays either fast convergence or the ability to handle transient engine operation.

2.6 Summary

The widespread use of alternative fuels is driven by the possible financial benefits as well as lower environmental impacts. Among the alternative fuels, natural gas has received special attention as its vast reserves can be found across the world. Unlike gasoline and diesel, natural gas can have different composition mainly depending on its location of extraction. In addition, local gas distributor companies make occasional changes to the pipeline gas composition which cause more variability.

The natural gas composition variation affects its combustion properties, such as stoichiometric air-fuel ratio, burn rate, lean-burn limits, and autoignition resistance. As a result, it poses a challenge for the conventional feedforward engine control strategy, which relies heavily on the static calibration maps acquired for one specific fuel composition. For example, the dependency of MBT spark timing for a CNG-fueled engine was demonstrated in [84]. This implies that calibrating the engine for one fuel composition will have penalties when a different compositions is used. In fact, a means of adaptation to the unknown fuel composition is necessary to ensure a fuel efficient operation.

On the other hand, ES as a control method for online optimization of unknown dynamical systems has been in development in control research community since 1922. The automotive research community has shown interest in the use of ES mainly to assist the lengthy manual calibration in design stage, favored by its independency of the engine model. Recently, motivated by the widespread of alternative fuels, ES has also been considered for online calibration of engines with variable fuel composition. However, this new application demands the ES to be fast which can not be achieved with the current implementations of ES. In addition it needs to cope with the transient nature of a typical
driving cycle required in automotive applications.

There are already existing results that address the slow convergence of ES. The model-based ES approach can potentially achieve shorter convergence time by warm starting the parameter estimator. However, it suffers from the three time-scale tuning that slows than the optimization process. In another approach, the fast black-box ES has been proposed that incorporates some information about the plant dynamics to allow an accelerated ES.

2.7 Research Aims

Reviewing the literature highlights two fundamental gaps in the existing research that will benefit from further research. The first of these is the development of a fast model-based ES technique that combines the advantages of a model-based approach and the accelerated performance of using a high dither frequency. In the context of the motivating applications in this thesis, this technique promises improved fuel economy by accelerating set-point calibration in stationary power plants and series hybrid vehicle applications. The second is the development of an ES technique that can track a time varying extremum caused by an exogenous disturbance. Again in the context of the motivating applications, these techniques promise improved calibration over a transient driving cycle for CNG-fueled vehicles. Therefore, this thesis pursues the following aims:

To develop a fast model-based ES scheme for Hammerstein plants

The “classical” implementations of ES require the closed-loop to occupy three separate time-scales, with the plant dynamics being the fastest, the estimator dynamics being slower, and the optimizer dynamics being the slowest. This can result in an undesirably slow optimization. A range of methods have been proposed to address this problem.

In the first approach, the plant steady-state map is assumed to be a known function of the input, but is parameterized by some unknown parameters. Hence, instead of estimating the gradient, an unknown parameter vector is estimated and used with a model-based optimization law to achieve ES. The parametrization is typically achieved through empirical or physical modeling. This approach, firstly, enables a broad range
of optimizers to be used in ES, and secondly, allows warm starting the parameter estimates with reasonable values based on some partially known properties of the function to improve the convergence time. In the literature, it is often referred to as “model-based ES” [108] or “ES for systems with parametric uncertainties” [36,89] or “grey-box ES” [85]. This method requires time-scale separation of the plant dynamics, parameter estimator and optimizer in a similar manner to the traditional black-box approaches [66,111], and consequently this tuning strategy generally exhibits slow convergence.

An alternative approach to address the slow convergence of ES is to avoid the three time-scale tuning of the closed loop altogether. More explicitly, time-scale separation of the plant and ES scheme is traditionally achieved by specifying the ES scheme to evolve in the $\Omega t$ time-scale, where $\Omega$ is a small dither frequency. To be able to use a high frequency dither signal, and potentially speed up ES convergence rate, some knowledge of the plant dynamics is needed. As discussed in the literature review section, existing results for fast ES typically treat the plant as having a Hammerstein [80] or a Wiener-Hammerstein structure [65,78]. This allows any nonlinearity within the plant to be represented within a static mapping which is sandwiched between Linear Time Invariant (LTI) input and output dynamics. Within this framework, it is possible to achieve arbitrarily fast semi-global convergence for black-box ES on both Hammerstein and Wiener-Hammerstein plants with fairly arbitrary static nonlinearities and requiring little more knowledge than the relative degree of the plants input and output dynamics. This is achieved through appropriate design of filters applied to the plants input and output.

In order to benefit from the advantages offered by both approaches, a fast model-based ES will be developed and its properties will be investigated. A rigorous proof will be provided for the proposed approach and the tuning requirements will be discussed.

To develop an ES scheme to accommodate time-varying extremum caused by measurable exogenous disturbance

In the majority of the ES algorithms, the underlying assumption is that the input-output map has a stationary input-output behavior. However, this assumption does not hold in some applications which may hinder the adoption of ES. The plant input–output be-
behavior may change when it is subject to external disturbances that may represent the operating point signals. Subsequently, the extremum point is time-variable.

A novel ES technique called “multiplexed ES” will be developed to estimate the optimal set-point across the range of the measurable exogenous disturbance. This will include the stability analyses and studies of the effect of the exogenous disturbance on a widely used black-box ES considered in [111].

**To demonstrate the performance of the proposed ES controllers in calibration of a CNG engine with unknown composition in steady state and transient operation**

The applied side of this research is divided into three parts.

- **First**, the proposed fast model-based technique will be tested for the steady state set-point calibration of spark timing and its performance will be investigated. This will include system identification to establish that the engine input–output dynamics can be regarded as a Hammerstein plant from spark timing to brake torque measurement.

- **Second**, the application of fast black-box ES introduced in [80] will be demonstrated for the steady state set-point calibration of injection duration with respect to the BSFC. This will include system identification to establish that the engine input–output dynamics can be represented by a Hammerstein-Wiener model from injection duration to BSFC. Since the result in [80] only applies to Hammerstein plants, a modification is needed to include Hammerstein-Wiener plants.

- **Lastly**, the proposed multiplexed ES scheme will be tested in online calibration of spark timing over a representative driving cycle.
Part I

Theory
Chapter 3

Fast Model-Based Extremum Seeking on Hammerstein Plants

This chapter proposes a fast model-based ES algorithm to benefit from the advantages offered by both high dither frequency and model-based approaches. A semi-global stability result is given for the proposed ES scheme acting upon a single-input single-output (SISO) Hammerstein plant. A Hammerstein plant model is established as a good model for many engineering systems, particularly for those where the dynamics are attributed to a slow sensor response [28, 59]. It is proven that the proposed ES scheme can achieve fast convergence rates by employing a sufficiently large dither frequency.

3.1 System Description

Figure 3.1 provides a schematic diagram of the proposed model-based ES controller. The goal of the proposed ES is to regulate the input as close to $u^*$ as possible. The ES loop consists of the plant followed by a filtering stage, a parameter estimator and an optimization law. A dither signal, $a \sin(\Omega t)$, is superimposed on the optimizer output to ensure identifiability of the unknown parameters in the plant map.

---

A substantial part of this chapter has been published in [102].

35
3.1.1 Plant

**Assumption 3.1.** The plant can be represented as a time-invariant SISO mapping, followed by a proper stable linear time invariant system \( F_o(s) \). The static map is considered to be a linear combination of basis functions \( \{q_1(u), q_2(u), \ldots, q_{n_\theta}(u)\} \). That is, the static map output can be represented as

\[
Q(u, \theta) = \sum_{i=1}^{n_\theta} q_i(u) \theta_i = q(u)^T \theta, \tag{3.1}
\]

where \( q(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n_\theta} \) is a vector of known basis functions and \( \theta \in \mathbb{R}^{n_\theta} \) is a vector of unknown parameters.

**Assumption 3.2.** The static map (3.1) has a unique minimum at \( u^* \). In addition, there exists \( v \in \mathbb{R}_{>0} \) such that the function basis \( q(u) \) is continuously differentiable sufficiently many times for any \( u \in [u^* - v, u^* + v] \).

The dependency of \( u^* \) on the unknown parameter, \( \theta \), is not shown explicitly, since \( \theta \) is assumed constant.

**Assumption 3.3.** The relative degree of the output dynamics \( F_o(s) \) is known.

3.1.2 Filter

The output filter \( F_1(s)G(s/\Omega) \) is designed to compensate the possible dither signal attenuation from the output filter, \( F_o(s) \), at high frequencies. If Assumption 3.3 holds, \( F_1(s)G(s/\Omega) \) can be designed such that \( F_o(s)F_1(s) \) is bi-proper and stable, and \( F_1(s)G(s/\Omega) \) is a proper stable filter. Section 3.3 demonstrates this concept for a given example.
3.1 System Description

Let \( F(s) := F_o(s)F_1(s) \). Without loss of generality \( F(\infty) = 1 \) (non-unity of \( F(\infty) \) can be incorporated in the definition of \( \theta \)). The following minimal state space representations are considered for \( F(s) \) and \( G(s/\Omega) \)

\[
\begin{align*}
\dot{x}_F &= \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} x_F \\ y_F \end{bmatrix}, \\
\dot{y}_F &= \begin{bmatrix} q(u)^T \theta \end{bmatrix}, \\
\end{align*}
(3.2a)
\]

\[
\begin{align*}
\dot{x}_G &= \begin{bmatrix} \Omega A_G & \Omega B_G \\ C_G & D_G \end{bmatrix} \begin{bmatrix} x_G \\ y_F \end{bmatrix}, \\
\end{align*}
(3.2b)
\]

where \( x_F \in \mathbb{R}^{N_F}, x_G \in \mathbb{R}^{N_G} \) and

\[
\begin{align*}
F(s) &= D_F + C_F(sI - A_F)^{-1}B_F, \\
G(s) &= D_G + C_G(sI - A_G)^{-1}B_G. \\
\end{align*}
(3.2c)
\]

3.1.3 Optimizer

The class of optimizer employed in the ES scheme is considered to take the following general form:

\[
\dot{\bar{u}} = -k\Omega F(\bar{u}, \hat{\theta}),
\]
(3.3)

where \( \hat{\theta} \) is the current estimate of \( \theta \), \( k\Omega \) is the optimizer gain, and \( F : \mathbb{R} \times \mathbb{R}^{n\theta} \rightarrow \mathbb{R} \). The following assumptions are placed on the optimization algorithm equation (3.3).

**Assumption 3.4.** The following hold when \( \hat{\theta} = \theta \) for all \( \bar{u} \in [u^* - v, u^* + v] \):

- the optimizer algorithm (3.3) has an asymptotically stable equilibrium at \( u^* \) corresponding to the unique minimum of \( q(\bar{u})^T \theta \);
- \( F(\bar{u}, \theta) \) is continuously differentiable and the Jacobian \( [\partial F / \partial \bar{u}] \) is bounded.

**Remark 3.1.** Let \( \bar{u} = \bar{u} - u^* \), then according to the converse Lyapunov theorem of [60], Assumption 3.4 necessitates the existence of \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K \) and a positive definite function \( V_u(\cdot) : \mathbb{B}^1(v) \rightarrow \mathbb{R} \), that satisfy the inequalities

\[
\begin{align*}
\alpha_1(\|ar{u}\|) \leq V_u(\bar{u}) \leq \alpha_2(\|ar{u}\|) \\
-\frac{\partial V_u}{\partial \bar{u}} F(\bar{u}, \theta) \leq -\alpha_3(\|ar{u}\|) \\
\end{align*}
(3.4a)
\]

\[
\begin{align*}
\alpha_4(\|ar{u}\|) \leq V_u(\bar{u}) \leq \alpha_5(\|ar{u}\|) \\
-\frac{\partial V_u}{\partial \bar{u}} F(\bar{u}, \theta) \leq -\alpha_6(\|ar{u}\|) \\
\end{align*}
(3.4b)
\]
\[ \left\| \frac{\partial V_u}{\partial \bar{u}} \right\| \leq \alpha_4(\bar{u}) \quad (3.4c) \]

**Remark 3.2.** One may select any optimizer that satisfies Assumption 3.4. A large number of potential optimization algorithms take the form of (3.3) satisfying Assumption 3.4, with examples including those discussed in [89]. The existence of such a Lyapunov function \( V_u(\bar{u}) \) can be satisfied implicitly through a converse Lyapunov method for any asymptotically stable optimizers.

**Assumption 3.5.** Let \( \tilde{\theta} = \hat{\theta} - \theta \). There exist \( (w, L_\theta) \in \mathbb{R}_{>0}^2 \) such that for any \( \bar{\theta} \in \mathcal{B}^w(w) \) the function \( F(\bar{u}, \theta + \bar{\theta}) \) is sector bounded in \( \bar{\theta} \) uniformly on \( \bar{u} \in [u^*-v, u^*+v] \)
\[
|F(\bar{u}, \theta + \bar{\theta}) - F(\bar{u}, \theta)| \leq L_\theta \|\bar{\theta}\|. \quad (3.5)
\]

**Remark 3.3.** Assumption 3.5 requires \( |F(\bar{u}, \theta + \bar{\theta}) - F(\bar{u}, \theta)| \) to be upper-bounded linearly by \( L_\theta \|\bar{\theta}\| \) on any given bound characterized by \( w \). Essentially, this assumption helps to correspond the bound on the value of the error in the estimation \( F(\bar{u}, \theta) \) using \( F(\bar{u}, \hat{\theta}) \), to the deviation of \( \hat{\theta} \) from \( \theta \). This may look like a restrictive assumption, since for example the Newton method may not satisfy the above assumption. However this situation can be rectified by using an appropriately modified Newton method such as that proposed in [81].

### 3.1.4 Parameter Estimator

By selecting sufficiently small \( k \), the output of the optimizer, \( \bar{u} \), is approximately stationary with respect to the signal, \( \sin(\Omega t) \), let \( \bar{q}(\cdot) \) represent the mean value of \( q(\cdot) \) over one period \( 2\pi/\Omega \):
\[
\bar{q}(\bar{u}) = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} q(\bar{u} + a \sin(\Omega t)) dt. \quad (3.6)
\]

Now, to generate a measure of the parameter estimation error, it is necessary to calculate an estimate of the output, \( y_G \), based on the current estimated parameter vector, \( \hat{\theta} \). The predicted output is represented as
\[
\hat{y}_G = \hat{y}_0 + \hat{y}_p^T \hat{\theta}. \quad (3.7a)
\]

where \( \hat{y}_p = G(s/\Omega)I_{n_\theta}[q(u) - \bar{q}(\bar{u})] \). In the estimation model (3.7a), \( \hat{y}_0 \) represents some “mean” value of the plant output, \( y_G \), and \( \hat{y}_p^T \hat{\theta} \) represents its zero-mean fluctuation. The
3.2 Stability Analysis

vector signal \( G(s/\Omega)I_{nu}[q(u) - \bar{q}(\bar{u})] \) is produced by passing the elements of \([q(u) - \bar{q}(\bar{u})]\) through an array of \( G(s/\Omega) \) filters. By using a state space representation of \( G(s/\Omega) \) in (3.2b) the following model for \( i \)th element of \( \hat{y}_p \) is obtained,

\[
\begin{bmatrix}
\dot{z}_{G,i} \\
\dot{\hat{y}}_{p,i}
\end{bmatrix} = \begin{cases}
\Omega A_G & \Omega B_G \\
C_G & D_G
\end{cases} \begin{bmatrix}
\dot{z}_{G,i} \\
[q_i(u) - q_i(\bar{u})]
\end{bmatrix},
\]

(3.7b)

where \( z_{G,i} \in \mathbb{R}^{N_G} \) is an internal state of the estimator. In this chapter, the gradient descent algorithm is used for the estimator. Let \( e := y_G - \hat{y}_G \) be the estimation error, using the gradient descent method leads to

\[
\begin{bmatrix}
\dot{\hat{y}}_0 \\
\dot{\hat{\theta}}
\end{bmatrix} = -\frac{\Omega}{2} \begin{bmatrix}
\partial^2 e / \partial \hat{y}_0^2 \\
\partial^2 e / \partial \hat{\theta}^2
\end{bmatrix} = \Omega \begin{bmatrix}
1 \\
y_p
\end{bmatrix} e.
\]

(3.8)

The parameter estimator is then implemented through (3.7b) and (3.8).

3.2 Stability Analysis

Superficially, the proposed ES scheme might be seen as an example of the model-based ES framework discussed in [89]. However, the result in [89] requires \( k \) and \( \Omega \) to be chosen sufficiently small; small \( k \) ensures the optimizer is slow compared to the estimator, and small \( \Omega \) ensures the estimator is slow compared to the plant dynamics. The tuning strategy proposed in this chapter is based on keeping the former of these time-scale separations while reversing the latter. In other words, \( \Omega \) is tuned to be sufficiently large so as to achieve accelerated estimation of the parameter vector \( \theta \) and subsequently accelerated optimization.

The static map output can be captured by the following Fourier series expansion

\[
q(\bar{u} + a \sin(\Omega t))^T \theta = \sum_{n \in \mathbb{Z}} \hat{q}_n(\bar{u}, a)^T \theta e^{i n \Omega t},
\]

(3.9a)
where
\[
\hat{q}_n(\bar{u}, a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(\bar{u} + a \sin(\tau)) e^{-in\tau} d\tau.
\] (3.9b)

The Fourier representation of \(q(u)^T \theta\) characterizes its periodic behavior when \(\bar{u}\) is held constant. Since \(q(u)^T \theta\) is the input to the linear output dynamics \(F(s)\), after applying linear systems theory, the steady-state response of the output dynamics for a given \(\bar{u}\) can be written as
\[
x^*_F(\bar{u}, a) = \sum_{n \in \mathbb{Z}} (in\Omega I - A_F)^{-1} B_F \hat{q}_n(\bar{u}, a)^T e^{in\Omega t}.
\] (3.10)

By letting \(\tilde{x}_F = x_F - x^*_F\), and substituting in (3.2a) it follows that
\[
y_F = C_F \tilde{x}_F + \sum_{n \in \mathbb{Z}} F(in\Omega) \hat{q}_n(\bar{u}, a)^T e^{in\Omega t}.
\] (3.11)

Importantly (3.11) holds even when \(\bar{u}\) is time-varying. \(y_F\) is the input to the next filter \(G(s/\Omega)\). The steady-state solution of \(x_G\) to this input can be calculated as
\[
x^*_G(\bar{u}, t) = -A_G^{-1} B_G C_F \tilde{x}_F + \sum_{n \in \mathbb{Z}} (in\Omega I - A_G)^{-1} B_G F(in\Omega) \hat{q}_n(\bar{u}, a)^T e^{in\Omega t}.
\] (3.12)

Now letting \(\tilde{x}_G = x_G - x^*_G\) and substituting into (3.2b) gives
\[
y_G = C_G \tilde{x}_G + G(0) C_F \tilde{x}_F + \sum_{n \in \mathbb{Z}} G(in) F(in\Omega) \hat{q}_n(\bar{u}, a)^T e^{in\Omega t}.
\] (3.13)

It is instructive to expand the plant output \(y_G\) in equation (3.13) further as
\[
y_G = G(0) C_F \tilde{x}_F + \sum_{n \in \mathbb{Z}} G(in) \hat{q}_n(\bar{u}, a)^T e^{in\Omega t} + C_G \tilde{x}_G.
\]
\[
+ \sum_{n \in \mathbb{Z}} G(in)(F(in\Omega) - 1) \hat{q}_n(\bar{u}, a)^T e^{in\Omega t}
\] (3.14)

Therefore, the output of the plant can be regarded as a summation of the following terms:

- The first two terms in equation (3.14) are “slow” terms; \(C_F \tilde{x}_F\) is the transient response of the output filters which is considerably slower than the dither signal, \(\sin(\Omega t)\), for sufficiently large \(\Omega\); \(\hat{q}_0(\bar{u}, a)^T \theta\) is a function of the optimizer output, \(\bar{u}\), which can be made arbitrary slow by choosing \(k\) to be small. The sum of these two terms will be estimated by \(\hat{y}_0\);
3.2 Stability Analysis

- The third term is a filtered version of the zero-mean periodic signal. This is the “useful” part of the output that can be used to estimate the unknown parameters \( \theta \);
- \( C_G \hat{x}_G \) is an error term associated with the fast filter. It will converge to a value which can be made small by tuning the parameters appropriately;
- The last two terms can be made arbitrarily small, since \( F(in\Omega) \to 1 \) for sufficiently large \( \Omega \).

Using the Fourier series expansion of \( q_i(u) \) (similar to (3.9a) for \( q(u) \)) in (3.7b), linear systems theory can be used to approximate the solution of the parameter estimator internal state, \( z_{G,i} \), by,

\[
\hat{z}_{G,i} = \sum_{n \in \mathbb{Z} \setminus \{0\}} (inI - A_G)^{-1} B_G \hat{q}_{n,i}(\bar{u}, a) e^{in\Omega t}.
\]  

(3.15)

By letting \( \tilde{z}_{G,i} = z_{G,i} - \hat{z}_{G,i} \), the exact solution for \( \hat{y}_{p,i} \) would be

\[
\hat{y}_{p,i} = C_G \hat{z}_{G,i} + \sum_{n \in \mathbb{Z} \setminus \{0\}} G(in) \hat{q}_{n,i}(\bar{u}, a) e^{in\Omega t}.
\]  

(3.16)

By defining the following error terms

\[
\tilde{y}_0 = \hat{y}_0 - G(0)(C_F \hat{x}_F + F(0)\overline{q(u)}^T \theta),
\]  

(3.17a)

\[
\tilde{\theta} = \hat{\theta} - \theta,
\]  

(3.17b)

\[
\tilde{u} = \bar{u} - u^*,
\]  

(3.17c)

and substituting (3.9a) and (3.10) into (3.2a), (3.11) and (3.12) into (3.2b), (3.9a) and (3.15) into (3.7b), (3.14) and (3.17a) and (3.17b) into (3.8), (3.17c) into (3.3) the closed loop error system is written as,

\[
\dot{x}_F = A_F \hat{x}_F + g_F,
\]  

(3.18a)

\[
\dot{x}_G = \Omega A_G \hat{x}_G + g_G,
\]  

(3.18b)

\[
\dot{z}_{G,i} = \Omega A_G \hat{z}_{G,i} + g_{z,i}, \quad \text{for } i = 1, 2, \ldots, n_\theta
\]  

(3.18c)

\[
\begin{bmatrix}
\dot{\tilde{y}}_0 \\
\dot{\tilde{\theta}}
\end{bmatrix} = -\Omega A_\theta \begin{bmatrix}
\tilde{y}_0 \\
\tilde{\theta}
\end{bmatrix} + g_\theta + e(\tilde{u} + u^*, a, \Omega t),
\]  

(3.18d)

\[
\dot{\tilde{u}} = -k\Omega F(\bar{u}, \theta) + g_u,
\]  

(3.18e)
where,

\[ g_F = -\frac{\partial x^*_F}{\partial \hat{u}} \hat{u}, \]  

\[ g_G = A_G^{-1} B_G C_F \hat{x} - \frac{\partial x^*_G}{\partial \hat{u}} \hat{u}, \]  

\[ g_{z,i} = -\frac{\partial z^*_{G,i}}{\partial \hat{u}} \hat{u}, \quad \text{for} \ i = 1, 2, \ldots, n_{\theta}, \]  

\[ A_{\theta}(\bar{u}, a, \Omega) = \begin{bmatrix} 1 & \hat{y}_p^T \\ \hat{y}_p & \hat{y}_p \hat{y}_p^T \end{bmatrix}; \]  

\[ g_{\theta} = \left[ -G(0)(C_F \hat{x} + F(0) \frac{\partial \hat{q}(\bar{u}, a)^T}{\partial \bar{u}} \hat{u}) \right] + \Omega \begin{bmatrix} 1 \\ \hat{y}_p \end{bmatrix} \left( C_G \bar{x} - \hat{y}_p^T \theta \right), \]  

\[ \hat{y}_p = [C_G \bar{z}_{G,1}; C_G \bar{z}_{G,1}; \ldots; C_G \bar{z}_{G,n_{\theta}}] \]  

\[ g_u = -k\Omega \left( F(\bar{u}, \theta + \hat{\theta}) - F(\bar{u}, \theta) \right) \]  

\[ e(\bar{u}, a, \Omega t) = \Omega \begin{bmatrix} 1 \\ \hat{y}_p \end{bmatrix} \sum_{n \in \mathbb{Z} - \{0\}} G(in)(F(in\Omega) - 1)\hat{q}_n^T(\bar{u}, a)e^{in\Omega t}. \]

The appealing feature of the above error system is that the dynamics of the estimator and the optimizer can be made arbitrarily fast by choosing a higher dither frequency. If the perturbation term \( e(\bar{u}, a, \Omega t) \) is ignored, then it can be shown that (3.18a)-(3.18e) has an equilibrium at the origin. Moreover, without interconnection terms \( g_F, g_G, g_{z,i}, g_{\theta} \) and \( g_u \), stability analysis of the remaining unperturbed isolated system is straightforward. The premise in this analysis is that each isolated subsystem is asymptotically stable, and the effect of interconnection and perturbation terms can be made small by tuning the parameters appropriately. For the isolated estimator, this requires the signal vector \([1; \hat{y}_p(\bar{u}, a, \Omega t)]\) in (3.18d) to fulfill a persistency of excitation criterion. According to Definition 3.4.1 in [49], if a signal vector \( r(t) \) satisfies the condition

\[ \frac{1}{2T} \int_{t}^{t+2T} r(\tau)r^T(\tau) d\tau \geq \alpha_0 I, \quad \forall t > t_0, \]

for some \((T, t_0, \alpha_0) \in \mathbb{R}_+^3\), it is “persistently exciting of level \( \alpha_0 \)”.

**Assumption 3.6.** For all \( \bar{u} \in \mathcal{B}^1(v) \) and for some \( m \geq n_{\theta} \), \( \Phi = [\hat{q}_1(\bar{u} + u^*, a) \hat{q}_2(\bar{u} + u^*, a) \ldots \hat{q}_m(\bar{u} + u^*, a)] \) is full rank.
Lemma 3.1. The signal vector \([1; \hat{y}_p(\bar{u}, a, \Omega t)]\) is persistently exciting of level \(O(a^{2m})\) iff Assumption 3.6 holds and both poles and zeros of \(G(s/\Omega)\) are in the open left half plane.

Proof. It can be shown that for a sufficiently small \(a\), \(\hat{q}_n\) is \(O(a^n)\). To prove Lemma 3.1, first note that the persistency of excitation criterion (3.20) can be investigated for the signal vector \([q - \hat{q}_0]\) by substituting its Fourier series representation for \(T = 2\pi/\Omega\). The integral will be a summation of rank-1 symmetric positive semi-definite matrices of the form \(\hat{q}^T_n \hat{q}_n\). If Assumption 3.6 holds, the aforementioned summation results in a full rank positive-definite matrix with the smallest eigenvalue of \(O(a^{2m})\). Next, by using Corollary 1 in [88], it can be shown that, since \([q - \hat{q}_0]\) is the input to the transfer function matrix \(G(s/\Omega)I_{n_a}\) and is persistently exciting, the output \(\hat{y}_p\) is persistently exciting as well. Note that \(G(s/\Omega)I_{n_a}\) is not dependent on \(a\), therefore it can be concluded that \(\hat{y}_p\) has the same PE level of \(O(a^{2m})\). It follows directly that if \(\hat{y}_p\) is persistently exciting, then \([1; \hat{y}_p]\) is persistently exciting of the same level.

Remark 3.4. The satisfaction of Assumption 3.6 can be investigated through evaluation of the smallest eigenvalue of

\[
\frac{1}{2\pi} \int_{t}^{t+2\pi} [q - \hat{q}_0][q - \hat{q}_0]^T(\bar{u}, a, \tau) d\tau.
\]

(3.21)

It is possible to find \(m\), by numerically investigating (3.21) in the \(a \to \infty\) limit and exploiting the fact that its smallest eigenvalue is \(O(a^{2m})\) from Lemma 3.1. See section 3.3.1 for an example.

Remark 3.5. When the nonlinear map \(q(\cdot)^T \theta\) is a polynomial with \(\theta\) representing the unknown coefficients, Assumption 3.6 can be satisfied with \(m = n_\theta\). Assume that the function basis is \(\{u^{n_\theta}, u^{n_\theta-1}, \ldots, u\}\). Fourier components \(\hat{q}_n\) are easily calculable for this function basis. It can easily be shown\(^1\) that the rank of \(\Phi\) equals the rank of

\[
\begin{bmatrix}
n_\theta \bar{u}^{n_\theta-1} & n_\theta(n_\theta - 1)\bar{u}^{n_\theta-2} & \ldots & n_\theta! \\
n_\theta - 1 \bar{u}^{n_\theta-2} & \vdots & \ldots & 0 \\
\vdots & 2 & \ldots & 0 \\
1 & 0 & \ldots & 0
\end{bmatrix},
\]

(3.22)

which is triangular, and therefore full rank for \(m = n_\theta\). Another example is given in section 3.3.

\(^1\)For example by using a Taylor series expansion of the vector signal \(q(\bar{u} + a \sin(\Omega t))\) around \(\bar{u}\) and noting that for polynomials the expansion has a finite number of terms.
According to Lemma 3.1, the first \( m \) Fourier components of \( q(u) \) must be linearly independent to be able to identify \( n_\theta \) parameters. It is reminiscent of the definition of a “sufficiently rich” signal in adaptive control literature, for example see Definition 3.4.2 and Theorem 3.4.2 in [49]. The following theorem is the main result of this chapter. To simplify the presentation here, \( k \) and \( \Omega \) are selected as

\[
k' = k/a^{2m} , \quad \Omega' = \Omega a^{2m} .
\]

(3.23)

**Theorem 3.1.** Consider the system described by (3.1)–(3.3) and (3.7b)–(3.8) under Assumptions 3.1–3.6. For any given \((r_F, r_G, r_z, r_y, r_\theta, r_u) \in \mathbb{R}_{>0}^4 \times (0, w) \times (0, \alpha_2^{-1} \circ \alpha_1(v)) \) and \( \mu \in \mathbb{R}_{>0} \) there exist \( a^*, k^*, \) and \( \Omega^* \) such that for all \((a, k', \Omega') \in (0, a^*) \times (0, k^*) \times (\Omega^*, \infty) \), any trajectory \((\tilde{x}_F(t), \tilde{x}_G(t), \tilde{z}_G, 1(t), \ldots, \tilde{z}_{G,n\theta}(t), \tilde{y}_0(t), \tilde{\theta}(t), \tilde{u}(t))\) originating in \( B_{N_F}^{N_F}(r_F) \times B_{N_G}^{N_G}(r_G) \times B_{N_G \times n_{\theta}}^{N_G \times n_{\theta}}(r_z) \times B^{u_0}(r_y) \times B^u_1(r_u) \) will satisfy,

\[
\limsup_{t \to \infty} \left\| \left[ \tilde{x}_F(t); \tilde{x}_G(t); \tilde{y}_0(t); \tilde{\theta}(t); \tilde{u}(t) \right] \right\|_{\infty} < \mu .
\]

Furthermore, there exist \( k^{**} \leq k^* \) such that for any fixed \((k', a) \in (0, k^{**}) \times (0, a^*) \) and \( \delta_u > \limsup_{t \to \infty} |\tilde{u}(t)| \), there is a \( T \in \mathbb{R}_{>0} \) such that \(|\tilde{u}(T)| < \delta_u \) for all \( t > T/\Omega \) and \( \Omega \geq \Omega^* \).

**Proof.** See Appendix A.

**Remark 3.6.** The first part of the theorem claims that starting from any initial condition where Assumption 3.1–3.6 hold, it is possible to achieve convergence of \( \tilde{u} \) to any desired close neighborhood of \( u^* \) through appropriate tuning of \( a, k, \) and \( \Omega \). Also, referring to the proofs in Appendix A reveals that \( \mu \) is \( O(ak') \) for exponentially stable optimizers (where Assumption 3.4 is strengthened).

**Remark 3.7.** The second part of Theorem 3.1 shows that under stronger assumptions than required in [89], the settling time of \( \tilde{u} \) is scaled with \( \Omega \) for sufficiently high dither frequency. Since according to Theorem 3.1, \( a, k', \) and \( \Omega' \) can be tuned independently, the higher rate of convergence of \( \tilde{u} \) (which is used here to define the term “fast” ES) is delivered through the use of a larger dither frequency. It should be noted that, in a real world scenario the dither frequency is bounded by practical considerations, such as noise and actuator bandwidth.

**Remark 3.8.** Unlike the black-box ES results presented in [78, 80], the proposed scheme allows a
broad range of optimizers to be used, such as gradient descent, Newton method, and Levenberg-Marquardt method. Additionally it is possible to aid the ES by “warm starting” the parameter estimator which can be useful in some applications. On the other hand, the present work requires dither frequency \( \Omega \in (\Omega^* a^{-2m}, \infty) \), whereas [80] also provides conditions for \( \Omega \) in some arbitrary range.

3.3 Simulation Example

The following simulation examples illustrate the performance of the proposed scheme. The application is inspired by the desire to suppress large pressure oscillations in a lean burn combustor [9], where a Hammerstein model is used to approximate the averaged characteristic of the underlying dynamics. The control input, \( u \), is the phase shift in the periodic fuel injection relative to the measured pressure signal. The underlying dynamics from the control input, \( u \), to the pressure oscillation amplitude, \( x \), is well-described by,

\[
\dot{x} = -b(x - h(u)),
\]

\[
h(u) = d + \sum_{i=1}^{3} \theta_{2i-1} \sin(i u) + \theta_{2i} \cos(i u),
\]

which has a Hammerstein structure, and \( d, \theta_1, \ldots, \theta_6 \) are the unknown parameters of the map. In order to validate the ES performance, the static map \( h(u) \) was obtained for one operating condition for which

\[
\theta = [\theta_1, \ldots, \theta_6] = [0.01416, -0.02029, 0.008214, -0.003017, 0.002992, 0.000787].
\]
The map has a minimum at \( u^* = -0.61 \) rad. Additionally, the decay rate \( b \) is assumed to be constant. For the purpose of simulation, \( b \) is picked to be 4. Thus the filtering stage can be developed as

\[
F_1(s) = s + 2, \quad G(s/\Omega) = \frac{1}{s/\Omega + 1}.
\]

In the following, two simulation examples are presented with two different optimizer algorithms for ES. The first example considers a gradient descent optimizer. The second example demonstrates how an optimizer may be developed that exploits the partially known plant model.

For simulations, the parameters of the ES loop are initialized as follows: \( \hat{\theta}_i = \theta_i + 0.01, x_F(0) \) and \( x_G(0) \) are chosen such that firstly \( y_G(0) = h(\bar{u}(0)) \) and secondly the filter output would not change if \( a, k = 0, \) and \( z_G(0) = 0. \) \( a \) is tuned to 0.4 rad and \( k = 1. \) For each approach, the optimizer output \( \bar{u} \) is presented for two different dither frequencies (40 and 200 rad/s) starting from two different initial conditions \( (\bar{u}(0) = 0.7 \) and \( \bar{u}(0) = -1.75 \) rad).

### 3.3.1 Gradient Descent Optimizer

Numerical experiments are performed to determine the value of \( m \) consistent with the definition in Assumption 3.6. To that end, note that the function basis can be stated as

\[
q(u) = [\sin(u); \cos(u); \sin(2u); \cos(2u); \sin(3u); \cos(3u)].
\]
For $u = \bar{u} + a \sin(\Omega t)$, the average (zero Fourier component) of this function basis can be calculated easily as

$$
\hat{q}_0(\bar{u}, a) = [J_0(a) \sin(\bar{u}); J_0(a) \cos(\bar{u}); J_0(2a) \sin(2\bar{u}); J_0(2a) \cos(2\bar{u}); J_0(3a) \sin(3\bar{u}); J_0(3a) \cos(3\bar{u})]
$$

where $J_0(\cdot)$ is a Bessel function of the first kind. Now substituting $q, \hat{q}_0$ in equation (3.21) reveals that for sufficiently small $a$, the persistency of excitation level $\alpha_0$ is $O(a^{12})$. Fig 3.3 depicts this fact for three different values $\bar{u}$. In other words Assumption 3.6 is met for $m = n_\theta = 6$. A gradient descent method is employed as the optimization algorithm, although any optimizer satisfying Assumptions 3.4 and 3.5 may be used.

One simulation result is presented which compares the effect of dither frequency on the convergence rate of the proposed scheme, as shown in Figure 3.4. As can be seen the results are almost overlaid on each other in the $\Omega t$ time-scale for the two frequencies, which implies that the convergence rate is proportional to the dither frequency used. In other words, for sufficiently large dither frequencies, a higher frequency would result in a proportionally faster convergence.

### 3.3.2 Model-based Optimizer

In this section another approach is taken towards the model-based ES. The current locally estimated map can be exploited to find the global extremum $u^*(\hat{\theta})$. The optimization law can be considered to be a feedback loop to regulate $\bar{u}$ to $u^*(\hat{\theta})$

$$
\dot{\bar{u}} = -k\Omega(\bar{u} - u^*(\hat{\theta})).
$$

In the same way that a Newton optimizer achieves an assignable exponential rate of convergence for a quadratic map, the optimizer used here achieves an assignable exponential rate of convergence regardless of the map. However, $u^*(\hat{\theta})$ is required to be analytically available.

For the sake of providing an example for a model-based optimizer, consider the case where the combustion instability map, $h(u)$, is given by the simpler model,

$$
 h(u) = d + \theta_1 \sin(u) + \theta_2 \cos(u), \quad u \in [-\pi, \pi].
$$
If the location of $u^*$ is known to be within $[-\pi/2, 0]$, the parameter estimator can be warm started with $\hat{\theta}_1(0) > 0$ and $\hat{\theta}_2(0) < 0$. In that case, the estimated minimum of (3.27) is obtained by

$$u^*(\hat{\theta}) = \tan^{-1}\left(\frac{\hat{\theta}_1}{\hat{\theta}_2}\right),$$

where $\tan^{-1}(\cdot) : \mathbb{R} \to [-\pi/2, \pi/2]$. To evaluate the performance of the proposed ES, a similar simulation study was carried out. In the simulations, equation (3.27) was used as the actual map. The optimizer output $\bar{u}$ is presented in Figure 3.5. Again the time-axis is in the $\Omega t$ scale and the results for the two frequencies are barely distinguishable, i.e. again the convergence rate increases with the dither frequency.

**Remark 3.9.** If no knowledge regarding the $u^*$ is available to warm start the parameter estimator, the following hybrid optimizer may be used

$$u^*(\hat{\theta}) = \begin{cases} 
\tan^{-1}\left(\frac{\hat{\theta}_1}{\hat{\theta}_2}\right) & \hat{\theta}_2 < 0 \\
\tan^{-1}\left(\frac{\hat{\theta}_1}{\hat{\theta}_2}\right) - \text{sign}(\hat{\theta}_1)\pi & \hat{\theta}_2 > 0 \\
-\text{sign}(\hat{\theta}_1)\frac{\pi}{2} & \hat{\theta}_2 = 0
\end{cases},$$

(3.29)

Note that, the definition of $u^*(\hat{\theta})$ has an inherent discontinuity at $(\hat{\theta}_1, \hat{\theta}_2) = (0, 0)$. As a result, Assumption 3.5 is not satisfied for all $w \in \mathbb{R}_{>0}$. In practice, this extension provides convergence for all initial conditions of the estimator (since it is highly improbable that $\hat{\theta}_1$ and $\hat{\theta}_2$ will simultaneously cross zero), although this result is not strictly guaranteed by the results in this chapter.

**Remark 3.10.** In practice, perfect model fits are likely to be difficult for real systems. Consequently, it is desirable for the ES scheme to exhibit some robustness to the plant–model mismatches. For example, application of (3.29) to the plant (3.24) demonstrates this property does hold. The simulation results shown in Figure 3.6 reveals that good convergence properties remain.
3.4 Conclusion

A fast model-based extremum seeking scheme for a Hammerstein plant was developed, where only the structure of the nonlinear map and the relative degree of the output filter need to be known. The proposed tuning strategy allows a wide range of optimizers to be used, and is able to achieve accelerated estimation and optimization of the steady-state behavior. The result is limited to Single Input-Single Output (SISO) Hammerstein plants. Two simulation examples were given to demonstrate the performance improvement achievable by employing the proposed “fast” extremum seeking. One of the proposed schemes demonstrated how more direct optimization algorithms can be implemented using knowledge of the map structure and parameters, without requiring calculation of map derivatives.
Chapter 4
Fast Extremum Seeking on a Class of Hammerstein-Wiener Plants

In this chapter, an extension of fast ES is developed for a special class of Hammerstein-Wiener plants, where the cost to be minimized is a known nonlinear function of the outputs of a mostly unknown Hammerstein plant.

4.1 Fast Extremum Seeking Controller Development

Consider the plant in Figure 4.1, where the desired cost metric is a function of two separate plant outputs, each of which is represented by a Hammerstein structure. The ES objective is to regulate \( u \) to a \( u^* \) that minimizes (or maximizes) \( j(\cdot) \) which is defined as

\[
 j(u) := j(q(u), h(u))
\]  

(4.1)

Assumption 2.5 now holds for both plant outputs. The functions \( q(u) \) and \( h(u) \) are unknown, and as a result \( j(u) \) is unknown (although \( j \) as a function of the plant outputs is known). \( F_o(s) \) and \( H_o(s) \) are internal dynamics, and \( \tilde{H}_o(s) \) and \( \tilde{F}_o(s) \) are additional filtering applied to the system outputs.

Assumption 4.1. Within the allowable input range defined by Assumption 2.6 the following holds

- \( q'(u^* + \sigma) \) and \( h'(u^* + \sigma) \) are continuous and bounded \( \forall \sigma \in [-v, v] \);

- \( j'(u^* + \sigma)/\sigma \geq K \) for all \( \sigma \in [-v, v] - \{0\} \).

A substantial part of this chapter has been published in [101].
Let $F(s) = F_o(s) \hat{H}_o(s)$ and $H(s) = H_o(s) \hat{F}_o(s)$. $F(s)$ and $H(s)$ have the following state space representations,

$$
\begin{align}
\dot{x}_F &= 
\begin{bmatrix}
A_F & B_F \\
C_F & D_F
\end{bmatrix}
\begin{bmatrix}
x_F \\
g(u)
\end{bmatrix}, \\
\dot{x}_H &= 
\begin{bmatrix}
A_H & B_H \\
C_H & D_H
\end{bmatrix}
\begin{bmatrix}
x_H \\
h(u)
\end{bmatrix},
\end{align}
$$

(4.2a)

(4.2b)

where $x_F \in \mathbb{R}^{N_F}$ and $x_H \in \mathbb{R}^{N_H}$. The input $v$ to the function $j(\cdot)$ in Figure 4.1 can be approximated as

$$
v = \bar{v} + \delta v + \mathcal{O}(a^2)
$$

where

$$
\bar{v} \approx f(\bar{u}), \quad \delta v = |F_o(i\Omega)|q'(\bar{u})a \sin(\Omega t + \varphi_F).
$$

Similarly, $w$ can be rewritten as $\bar{w} + \delta w + \mathcal{O}(a^2)$. By this interpretation of $v$ and $w$, the output can be expressed as

$$
j(v, w) = j(\bar{v}, \bar{w}) + \frac{\partial j}{\partial v} \delta v + \frac{\partial j}{\partial w} \delta w + \mathcal{O}(a^2)
$$

(4.3)

where the partial derivatives are evaluated at $(\bar{v}, \bar{w})$. Referring to the definition of the observer in (2.12), $\dot{x}_2$ is expected to converge to the first harmonic of the system output, $y$, at frequency $\Omega$,

$$
\frac{\partial j}{\partial v} |F(i\Omega)|q'(\bar{u})a \sin(\Omega t + \varphi_F) + \frac{\partial j}{\partial w} |H(i\Omega)|h'(\bar{u})a \sin(\Omega t + \varphi_H).
$$

---

1See equation (12) in [80] for exact derivation.
This is required to be proportional to the \( \frac{\partial j}{\partial u} \) which is

\[
\frac{\partial j}{\partial v} q'(\bar{u}) + \frac{\partial j}{\partial w} h'(\bar{u})
\]  

(4.4)

To that end, ideally the following assumption should hold.

**Assumption 4.2.** For all \( \Omega \in D \) the following holds:

(a) \( |F(i\Omega)| = |H(i\Omega)| \);

(b) \( \arg F(i\Omega) = \arg H(i\Omega) \).

**Remark 4.1.** To satisfy Assumption 4.2 \( \hat{H}_o(s) \) and \( \hat{F}_o(s) \) have to be designed by exploiting the knowledge of \( F_o(s) \) and \( H_o(s) \) at frequency \( \Omega \).

If Assumption 4.2 holds the observer states are given by

\[
\hat{x} = \begin{bmatrix}
\hat{j}_0 \\
\text{aIm}\{e^{i\Omega t} F(i\Omega)\} j'(\bar{u}) \\
\text{aRe}\{e^{i\Omega t} F(i\Omega)\} j'(\bar{u})
\end{bmatrix} + \mathcal{O}(a^2).
\]  

(4.5)

If one of the conditions in Assumption 4.2 does not hold, the gradient estimate may be subject to an offset that would result in the ES scheme converging to a non-optimal input. A similar type of behavior is discussed in [115]. In light of (4.2), Assumption 2.8 is modified to the following:

**Assumption 4.3.** There exist \( J \in \mathbb{R}_{>0} \) and \( D \subset \mathbb{R}_{>0} \) such that for all \( \Omega \in D \)

\[
\text{Re}\{\exp(i\phi(\Omega)) F(i\Omega)\} \geq J.
\]  

(4.6)

**Theorem 4.1.** Consider (4.1),(4.2a-4.2b) and (2.12) under Assumptions 2.5,2.7,4.1–4.3. For any \( (\beta_{x_F}, \beta_{x_H}, \beta_x) \subset \mathbb{R}_{N_F} \times \mathbb{R}_{N_H} \times \mathbb{R}^3 \) and \( r_u \in (o, r) \), there exist \( (k^*, a^*, c) \in \mathbb{R}_3^+ \) such that, for all \( (k, a, \Omega) \in (0, k^*) \times (0, a^*) \times D \), any trajectory with initial condition \( (x_F, x_H, \hat{x}, \bar{u} - u^*) \in \beta_{x_F} \times \beta_{x_H} \times \beta_x \times (-r_u, r_u) \) will asymptotically converge to a \((2\pi/\Omega)\)-periodic solution satisfying,

\[
\lim_{t \to \infty} |\bar{u} - u^*| \leq cka.
\]  

(4.7)

**Proof.** The proof follows a similar approach as in [80].
Remark 4.2. Note that the structure shown in Figure 4.1, also includes a class of Hammerstein-Wiener plants where the output is defined as \( j(v) \). Similarly, it is trivial to extend this work for plants where the cost functional depends upon an arbitrary number of plant outputs where each output is described by a static nonlinearity with LTI output dynamics.

4.2 Simulation

Consider the plant structure shown in Figure 4.1 with the following characteristics:

\[
q(u) = -12.6534u^2 + 165.6649u - 454.3996, \\
F_0(s) = \frac{0.04533s^4 + 2.364s^3 + 237s^2 + 3805s + 1.606e05}{s^4 + 28.18s^3 + 1366s^2 + 1.905e04s + 1.606e05} \\
h(u) = 8.275u - 12.894, \quad H_o(s) = 1,
\]

and with a cost metric function of:

\[
j(v, w) = v/w.
\]
In fact, this plant approximates the input-output behavior of a test engine as a plant with the injection duration as input and the Brake Specific Fuel Consumption (BSFC) as performance metric. It can be shown that at steady state, \( j(\cdot) \) has a minimum at \( u^* = 5.8 \) ms/cycle. Two simulation examples are given to demonstrate the performance of the proposed ES scheme. First, the performance of the proposed controller is demonstrated for one choice of compensator. Then, different choices of the compensator will be investigated. For all simulations the dither signal amplitude was \( a = 0.1 \) ms/cycle, and the optimizer gain was \( k = 10 \).

In the first simulation example, \( F_o(s) \) is assumed known to the designer. Therefore, the compensator is designed with \( \hat{F}_o(s) = F_o(s) \) and \( \hat{H}_o(s) = 1 \), which satisfies Assumption 4.2 for all \( \Omega \in \mathbb{R}_{>0} \). The ES performance is shown in Figure 4.2 for dither frequency of 10, 100, and 1000 rad. Notably, the results are almost overlaid on each other in the \( \Omega t \) time-scale for the large dither frequencies of 100 and 1000 rad, which implies that the convergence rate is proportional to the dither frequency used.

A simulation study was conducted to investigate different choices for \( \hat{F}_o(s) \) while \( \hat{H}_o(s) = 1 \). The following different choices reflecting different level of knowledge of \( F_o(s) \) were tested:

- \( \hat{F}_{o,1}(s) = 1 \).
- \( \hat{F}_{o,2}(s) = \exp(-0.122 s) \).
- \( \hat{F}_{o,3}(s) = F_o(s) \)

As shown in Figure 4.3, \( \hat{F}_{o,2}(s) \) only compensates for the phase shift effect of \( F_o(s) \). Figure 4.4 shows the simulation results for the three cases discussed above in terms of the ES output offset from the \( u^* \). It can be seen that solely compensating for phase shift by \( \hat{F}_{o,2}(s) \) can achieve ES for frequencies up to 1 Hz and as frequency increases an offset appears in the final value of the optimizer output, \( \bar{u}_\infty \). When \( \hat{F}_o(s) = F_o(s) \), \( |\bar{u}_\infty - u^*| \) is equal to zero, but that means more knowledge about the plant dynamics is exploited. The simulation results demonstrates that with more accurate knowledge of the dynamics, a more informed decision can be made for the compensation as might be reasonably expected.
4.3 Conclusion

A fast ES algorithm was developed to achieve fast convergence rate through careful exploitation of the plant dynamics knowledge for a class of Hammerstein-Wiener plants. It was shown that when the phase shift effects of the plant input–output dynamics are known, a compensator stage can be designed that allows the use of the desired frequency range. Also, it was shown that, different level of knowledge about the input–output dynamics can be used in designing the compensator stage, which leads to different level of ES accuracy.
Chapter 5
Multiplexed Extremum Seeking for Systems with Exogenous Inputs

Most extremum seeking (ES) design approaches consider optimizing the steady state output of a plant with stationary input-output behavior. However, this setting is not applicable to the large class of systems that exhibit a time-varying steady state input-output mapping caused by exogenous input disturbances, \( w \). In essence, this exogenous input may require the steady state extremum in the typical extremum seeking formulations, \( u^* \), to be replaced with an extremum mapping, \( u^*(w) \). While the “fast” ES techniques discussed in Chapter 3 and 4 can achieve accelerated performance, the results do not cover the presence of a varying (possibly rapidly) disturbance signal. To cope with this, a multiplexed ES scheme is proposed in this chapter.

The contributions of this chapter is twofold. First, a rigorous stability analysis is presented to investigate the effect of exogenous inputs on the behaviour of the widely used black-box ES introduced in [66, 111]. The main result of the first part gives the sufficient condition under which ES output will converge close to an “average” extremum which lies in the locus of all possible extremums produced by the exogenous inputs. This property can be exploited by dividing the domain of the exogenous inputs and employing multiple independent ES agents in each subdomain. Hence, a multiplexed ES framework is proposed consisting of multiple ES agents that are individually activated based on the exogenous input. The proposed approach is shown to guarantee improved accuracy under specific condition. The approach is demonstrated via simulations to achieve a finer estimation of the extremum map, thereby implying better overall dynamic performance.

A substantial part of this chapter has been submitted as [103].
5.1 System Description

5.1.1 Plant

Consider a nonlinear plant of the following form

\[ \dot{x} = f(x, u, w(t)) \tag{5.1a} \]
\[ y = h(x, u) \tag{5.1b} \]

where \( f : \mathbb{R}^{N_x} \times \mathbb{R} \times \mathbb{R}^{N_w} \rightarrow \mathbb{R}^{N_x} \) and \( h : \mathbb{R}^{N_x} \times \mathbb{R} \rightarrow \mathbb{R} \). \( u \) and \( w \) denote the control input and the set of measurable exogenous disturbances. It is assumed that \( w \in \Gamma \) for some compact set \( \Gamma \subset \mathbb{R}^{N_w} \). To simplify the notations in this chapter it is assumed that \( \Gamma \) is upper and lower bounded on each dimension as,

\[ \Gamma = \{ w | w_j \leq w_j \leq \bar{w}_j, \text{ for } j = 1, \ldots, N_w \}, \tag{5.2} \]

for some \( \{ \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{N_w} \} \).

The following assumptions are considered on the plant and the disturbance signals:

**Assumption 5.1.** \( h, f, f'_x, f'_u, \) and \( f'_w \) are continuous on \( \mathbb{R}^{N_x} \times \mathbb{R} \times \mathbb{R}^{N_w} \).

**Assumption 5.2.** There exists \( \epsilon_1 \in \mathbb{R}_{>0} \) such that \( \| \dot{w}(t) \| \leq \epsilon_1 \).

**Remark 5.1.** \( w(t) \) is considered a slowly varying input for (5.1a) if \( \epsilon_1 \) is sufficiently small.

**Assumption 5.3.** There exists a twice continuously differentiable function \( S : \mathbb{R} \times \mathbb{R}^{N_w} \rightarrow \mathbb{R}^{N_x} \) such that \( f(x, u, w) = 0 \), if and only if \( x = S(u, w) \). Moreover this solution is asymptotically stable uniformly in \( u \) and \( w \) and for all \( (x, u, w) \in \mathbb{R}^{N_x} \times \mathbb{R} \times \Gamma \).

**Remark 5.2.** Assumption 5.3 is an extension of Assumption 2 of [111] in order to account for the effect of the exogenous input on the solution of the system.

Let \( Q(u, w) := h(S(u, w), u) \) denote the steady-state input-output map for each fixed values of \( u \) and \( w \).

**Assumption 5.4.** There exists a smooth function \( u^*(w) : \mathbb{R}^{N_w} \rightarrow \mathbb{R} \) such that for all \( w \in \Gamma, \)

\[ Q'_u(u^*(w), w) = 0. \tag{5.3} \]

---

\(^1\)Throughout this chapter the minimization problem is considered without loss of generality.
Assumption 5.5. There exists $b \in \mathbb{R}_{>0}$ such that for all $w \in \Gamma$,
\[ b\zeta^2 < Q'(\zeta + u^*(w), w)\zeta, \quad \forall \zeta \neq 0. \] (4)

Assumptions 5.4 and 5.5 state that $u^*(w)$ is the minimizer of $Q(u, w)$ with respect to $u$ for a given $w$. Therefore, the objective of the controller design is to shift the input $u$ close to $u^*(w)$ while the system is subject to a time-varying exogenous input signal vector $w(t)$. Before moving on, the following example is provided to clarify the type of problems considered in this chapter.

Example 5.1. Suppose that the state-space representation of a dynamical system is described by
\[
\dot{x} = Ax + BQ(u, w), \\
y = Cx + DQ(u, w),
\]
where $x \in \mathbb{R}^{N_x}$ are the plant states, $u \in \mathbb{R}$ and $w \in \mathbb{R}$ are the controlled input and the exogenous input respectively, and $Q(u, w) = u^2 - wu$. It follows that for fixed $w$, the input-output map at steady-state, $Q(u, w)$, has a local minimum at $u^* = w/2$ with respect to $u$. It is emphasized that when $w$ is time varying, i.e. $\|\dot{w}\| \neq 0$, the optimal input is not precisely located at $w/2$. However, for sufficiently small $\|\dot{w}\|$ with respect to the plant dynamics it stays close to $w/2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{es_scheme.png}
\caption{ES scheme influenced by an exogenous input signal vector $w(t)$.}
\end{figure}

5.2 Stability Analysis

The main result of this section is concerned with the stability analysis of a class of black-box ES schemes that is studied in [66,111] applied on a dynamical plant perturbed with slowly varying exogenous inputs.
Consider the ES feedback loop shown in Figure 5.1. In this scheme, $\bar{u}$ represents the current estimation of $u^*(w)$. The state equation describing the feedback loop is stated as,

\begin{align}
\dot{x} &= f(x, u, w(t)), \\
\dot{\bar{u}} &= -k\Omega h(x, u) \sin(\Omega t), \\
u &= \bar{u} + a \sin(\Omega t),
\end{align}

where $\Omega$ is the dither signal frequency, $a$ is the dither signal amplitude, and $k$ is the adaptation gain.

The following stability analysis is presented in three parts. In the first part, the behavior of a perturbed gradient system is studied. It is followed by the analysis of a static map ES scenario perturbed by exogenous input disturbances. It shows the condition under which the ES dynamics can be regarded as a perturbed gradient system in an averaged sense. In the third part, the main result regarding the stability of the ES scheme applied to a dynamic plant with exogenous inputs is presented.

5.2.1 Preliminary Analysis for a Perturbed Gradient System

As the first step in the analysis, it is convenient to consider the behavior of a perturbed gradient system. Suppose that the function $Q(u, w(t))$ is known and it satisfies Assumption 5.4 and 5.7. The gradient system is then given by

\begin{equation}
\dot{\bar{u}} = -kQ'_u(u, w(t)), \quad w(t) \in \Gamma,
\end{equation}

where $Q'_u$ denotes the gradient of $Q$ with respect to $u$, and $k \in \mathbb{R}_{>0}$ is the gradient descent gain. To show stability for this gradient system, the general averaging theorem can be employed. Given the continuity of $Q'_u(u, w)$ in $w$, the mean-value theorem\(^2\) can be used to show that for any fixed $u$ there exists a $\bar{w} \in \Gamma_w$ such that

\begin{equation}
Q'_u(u, \bar{w}) = \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} Q'_u(u, w(\tau))d\tau.
\end{equation}

\(^2\)The first mean-value theorem states that if $f(x)$ is continuous in $x$ over $[a, b]$ then there exists a $c \in (a, b)$ such that $\int_{a}^{b} f(x) = f(c)(b - a)$. 


This can be easily shown, by taking $Q'_u(u, w(\tau))$ as a general function of $\tau$. Then mean-value theorem can be used to conclude that there exists a $\tau^0 \in (t, t + T)$ such that \[
\frac{1}{T} \int_{t}^{t+T} Q'_u(u, w(\tau))d\tau = Q'_u(u, w(\tau^0)).\] In (5.6) $\bar{w} = w(\tau^0)$.

Next, considering the continuity of $Q'_u(u, w(t))$ and the definition of $Q'_u(u, \bar{w})$ in (5.6), it can be shown that there exists a positive constant $K$ and a class $\mathcal{L}$ function $\sigma(\cdot)$ such that, \[
\left\| \frac{1}{T} \int_{t}^{t+T} Q'_u(u, w(\tau))d\tau - Q'_u(u, \bar{w}) \right\| \leq K\sigma(T). \tag{5.7}
\]

According to the general averaging theorem in [60], it follows that $Q'_u(u, \bar{w})$ can be considered as an “average” value of $Q'_u(u, w(t))$. By the same token, $Q(u, \bar{w})$ is referred to as an average value of $Q(u, w(t))$. In fact equation (5.6) defines the value of $\bar{w}$ which has a central role in the stability analysis in this chapter. It allows to study stability of the following system instead of (5.5) \[
\dot{u}_{av} := -kQ'_u(u_{av}, \bar{w}). \tag{5.8}
\]

More rigorously, if $u^*(\bar{w})$ denotes the extremum of $Q(u_{av}, \bar{w})$, then by letting $\bar{u}_{av} = u_{av} - u^*(\bar{w})$, state equations (5.8) can be transformed into its error coordinates as, \[
\dot{\bar{u}}_{av} := -kQ'_u(\bar{u}_{av} + u^*(\bar{w}), \bar{w}). \tag{5.9}
\]

Now the general averaging results of [60] can be employed to establish that if $\bar{u}_{av}$ asymptotically converges to zero, for sufficiently small $k$, $u$ will converge to a $\mathcal{O}(k)$ neighbourhood of $u^*(\bar{w})$ as well. In other words, the fact that $k$ can be made small is exploited to conclude that the perturbed gradient system behaves like an unperturbed gradient system in an “averaged” sense. It is also instructive to note that when $k \to 0$, $u$ will converge on average to the close vicinity of the extremum point of $Q(u, \bar{w})$.

### 5.2.2 Analysis for a Static Map Extremum Seeking

In the second step the ES behavior for an unknown static plant is investigated. The analysis in this subsection allows the effect of an exogenous input signal on the ES performance to be considered without dealing with the complication of plant dynamics.
Suppose that the input-output mapping $Q(u, w(t))$ is unknown and that there are no plant dynamics. As before, Assumptions 5.4 and 5.7 still hold, $u$ is the controlled input and $w(t)$ is an exogenous input. The state equation (5.4a)-(5.4c) for the feedback loop can be rewritten as,

\[
\begin{align*}
y &= Q(u, w(t)), \quad w(t) \in \Gamma \\
u &= \bar{u} + a \sin(\Omega t), \\
\dot{\bar{u}} &= -k \Omega Q(u, w(t)) \sin(\Omega t).
\end{align*}
\]

(5.10a)-(5.10c)

To study stability, consider $u^*(\bar{w})$ as the minimum point of the static map $Q(u, \bar{w})$ for some fixed $\bar{w} \in \Gamma$ which is yet to be defined. Let $\tilde{u} := \bar{u} - u^*(\bar{w})$. Using the new error coordinate, (5.10c) can be transformed into:

\[
\dot{\tilde{u}} = -k \Omega [Q(\tilde{u} + u^*(\bar{w}), w(t)) \sin(\Omega t)].
\]

(5.11)

Taylor series expansion of $Q(\cdot)$ on its first argument around $\tilde{u} + u^*(\bar{w})$ can be used to restate (5.11) further as,

\[
\dot{\tilde{u}} = -k \Omega [f_1(\tilde{u}, t, \Omega t) + a f_2(\tilde{u}, t, \Omega t) + O(a^2)],
\]

(5.12)

where

\[
\begin{align*}
f_1(\tilde{u}, t, \Omega t) &= Q(\tilde{u} + u^*, w(t)) \sin(\Omega t), \\
f_2(\tilde{u}, t, \Omega t) &= Q'_u(\tilde{u} + u^*, w(t)) \sin^2(\Omega t).
\end{align*}
\]

In equation (5.12), the last term is the remainder of the expansion and is negligible for sufficiently small $a$. Next, the following assumption is useful to find an average for the first two terms of (5.12),

**Assumption 5.6.** For any $\bar{u} \in \mathbb{R}$, $w(t) \in \Gamma$ and for all $t \geq t_0$,

\[
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} Q(\bar{u}, w(\tau)) \sin(\Omega \tau) d\tau = 0, \\
\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} Q'_u(\bar{u}, w(\tau)) \cos(2\Omega \tau) d\tau = 0.
\]

(5.14a)-(5.14b)

**Remark 5.3.** Assumption 5.6 states that the dither signal and the output must not be correlated,
which in turn implies that the dither signal and the exogenous input must be not be correlated. The following examples demonstrate that it is not too restrictive:

- If \( w(t) \) is periodic with the fundamental period \( T_w \neq 2\pi/\Omega \), then Assumption 5.6 is satisfied. Furthermore there exists \( \Omega^* \in \mathbb{R}_{>0} \) such that for all \( \Omega \in (0, \Omega^*) \) Assumption 5.6 holds.

- For any arbitrary signal \( w(t) \) with zero content at \( \Omega \) rad/s, Assumption 5.6 is satisfied.

- For any constant \( w(t) \), i.e. \( w(t) = c \) for some \( c \in \mathbb{R}^{N_w} \), Assumption 5.6 is satisfied. Indeed, this will transform the ES problem back to the one considered in [111].

**Remark 5.4.** The exogenous input is often not a controlled input, therefore to satisfy Assumption 5.6 some level of knowledge must exist about \( w(t) \) a priori.

If Assumption 5.6 holds, then one can define an average for the first two terms of the RHS of (5.12) as,

\[
\begin{align*}
    f_{av}(\tilde{u}, \bar{w}) := & \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} f_1(\tilde{u}, \tau, \Omega \tau) + a f_2(\tilde{u}, \tau, \Omega \tau)d\tau = \frac{a}{2} Q_u'(\tilde{u} + u^*, \bar{w}),
\end{align*}
\]

(5.15)

where \( \bar{w} \in \Gamma \). Just as in (5.6), (5.15) defines \( \bar{w} \) and subsequently the average extremum \( u^*(\bar{w}) \) used previously in defining \( \tilde{u}_{av} \). The autonomous average system is then introduced as,

\[
\begin{align*}
    \dot{\tilde{u}}_{av} = & -k \Omega \frac{a}{2} Q_u'(\tilde{u}_{av} + u^*(\bar{w}), \bar{w}).
\end{align*}
\]

(5.16)

To see in what sense the trajectories of (5.10c) are close to that of (5.16), the general averaging result in Chapter 10.6 of [60] is of use. In fact by introducing a proper transformation the error system (5.10c) can be regarded as the averaged system (5.16) plus some perturbation terms. The details of the steps are presented in Appendix B.1. As shown in the appendix, a near-identity transformation is introduced as,

\[
\begin{align*}
    \tilde{u}(t) := & s(t) + k \Omega p(t, s(t), k),
\end{align*}
\]

(5.17)

where \( p(\cdot) \) is a bounded term and defined in equation (B.2) in Appendix B.1. Following the arguments in Appendix B.1, the state equation for \( s(t) \) is obtained as,

\[
\begin{align*}
    \frac{ds}{dt} = & -k \Omega \frac{a}{2} Q_u'(\tilde{u} + u^*, \bar{w}) + O(k \Omega^2 \alpha(k)) + O(ka^2 \Omega),
\end{align*}
\]
where \(\alpha(\cdot)\) is a class \(K\) function. This expression for \(s(t)\) dynamics reveals that it evolves in the gradient direction of the averaged system, \(Q(\cdot, \bar{w})\), with some bounded perturbation. Furthermore the effect of the perturbation terms can be made arbitrarily small by tuning each of \((a, k, \Omega)\) sufficiently small. On the other hand, as shown in Appendix A, if \(s(t)\) remains bounded \(\tilde{u}(t)\) will stay within \(O(\Omega \alpha(k))\) of \(s(t)\). Hence, it implies that (5.11) will shift \(\tilde{u}(t)\) to the close vicinity of \(u^*(\bar{w})\) in an averaged sense.

5.2.3 Analysis for a Dynamical Plant

Now consider the full closed-loop system, (5.4a)-(5.4c). In order to study stability for this system the following transformations are introduced,

\[
\tilde{x} := x - S(u, w(t)), \quad (5.18a) \\
\tilde{u} := \bar{u} - u^*(\bar{w}), \quad (5.18b)
\]

for \(\bar{w} \in \Gamma\) defined by (5.15). The state equations for these new coordinates are,

\[
\dot{\tilde{x}} = f(\tilde{x} + S, u, w(t)) - S'_u[\dot{\tilde{u}} + a\Omega \cos(\Omega t)] - S'_w \dot{w} \\
\dot{\tilde{u}} = -k\Omega h(\tilde{x} + S, u) \sin(\Omega t). \quad (5.19b)
\]

The following theorem presents the main result of this section.

**Theorem 5.1.** Suppose that Assumptions 5.1-5.6 hold. Then for the system given by (5.19a)-(5.19b), for any \((\delta_x, \delta_u, \mu) \in \mathbb{R}^3_{>0}\) there exist \((a^*, \Omega^*) \in \mathbb{R}^3_{>0}\) such that for any \(a \in (0, a^*)\) there exist \(k^*\) and \(\gamma(\cdot) \in K\) such that for all \((k, \Omega) \in (0, k^*) \times (0, \Omega^*)\), any trajectories \((\tilde{u}(t), \tilde{x}(t))\) originating in \(B(\delta_u) \times B^{N_x}(\delta_x)\) will satisfy

\[
\lim_{t \to \infty} \|[(\tilde{x}(t); \tilde{u}(t))]\| < \mu + \gamma(\epsilon_1).
\]

**Proof:** See Appendix B.2 for the proof.

**Remark 5.5.** The stability analysis is done by combining the general averaging results in [60] and the Input-State practical Stability (ISpS) results in [114]. In that regards it is inspired by the approach taken in [111] for proving non-local stability of black-box ES in the absence of exogenous inputs. This is appealing, because in the presence of an exogenous input, the problem cannot be
5.2 Stability Analysis

CAST INTO THE STANDARD FORM REQUIRED BY THE EXISTING SINGULAR PERTURBATION RESULTS USED IN ES ANALYSIS. HOWEVER THE NOTION OF TIME-SCALE SEPARATION STILL EXISTS IN THE TUNING STRATEGY PRESCRIBED BY THE MAIN RESULT OF THIS CHAPTER.

ASSUMPTION 5.5 CAN BE RESTRICTIVE FOR SOME APPLICATIONS. A WEAKER RESULT IS STILL POSSIBLE UNDER LESS RESTRICTIVE ASSUMPTIONS. FOR INSTANCE, INSTEAD OF REQUIRING ASSUMPTION 5.5 TO HOLD GLOBALLY, THE FOLLOWING REGIONAL ASSUMPTION COULD BE USED:

ASSUMPTION 5.7. THERE EXISTS \((b, v) \in \mathbb{R}^2_+\) SUCH THAT FOR ALL \(w \in \Gamma\)

\[
bc^2 < Q'((\zeta + u^*(w), w)\zeta, \zeta \in B^{N_u}(v) - \{0\}).
\]

(5.20)

THE FOLLOWING THEOREM SUBSEQUENTLY EXTENDS THEOREM 5.1 TO A SEMI-GLOBAL STABILITY RESULT BASED ON ASSUMPTION 5.7.

THEOREM 5.2. SUPPOSE THAT ASSUMPTIONS 5.1-5.4, 5.6, 5.7 HOLD. THEN FOR THE SYSTEM GIVEN BY (5.19a)-(5.19b), FOR ANY STRICTLY POSITIVE \(\delta_u, \mu\) WITH \(\delta_u \in (0, v)\) THERE EXIST \((a^*, \epsilon_1^*, \Omega^*) \in \mathbb{R}^3_+\) SUCH THAT FOR ANY \(a \in (0, a^*)\) THERE EXIST \(\delta_x, k^*\) AND \(\gamma(\cdot) \in K\) SUCH THAT FOR ALL \((k, \Omega) \in (0, k^*) \times (0, \Omega^*)\) AND \(\epsilon_1 \leq \epsilon_1^*\), ANY TRAJECTORIES \((\tilde{u}(t), \tilde{x}(t))\) ORIGINATING IN \(B(\delta_u) \times B^{N_x}(\delta_x)\) WILL SATISFY

\[
\lim_{t \to \infty} ||[\dot{x}(t); \tilde{u}(t)]|| < \mu + \gamma(\epsilon_1)
\]

(5.21)

PROOF: SEE APPENDIX B.3 FOR THE PROOF.

REMARK 5.6. THEOREM 5.2 DEMONSTRATES THAT FOR THE FEEDBACK LOOP TO BE PRACTICALLY ASYMPTOTICAL STEABLE, THE TUNING OF THE DITHER AMPLITUDE \(a\) DETERMINES TWO OTHER PARAMETERS: 1) THE MAXIMUM OF THE ADAPTATION GAIN \(k^*\); 2) THE MAXIMUM TOLERABLE INITIAL CONDITION BOUND FOR THE PLANT STATES \(\delta_x\).

REMARK 5.7. IN THEOREM 5.1 AND 5.2, \(\epsilon_1\) CONSTITUTES ONE PART OF THE ULTIMATE BOUND. BUT UNLIKE THEOREM 5.1, IN THEOREM 5.2 \(\epsilon_1\) IS TREATED AS A TUNING PARAMETER, ALTHOUGH TYPICALLY IT IS NOT AVAILABLE FOR TUNING. IN FACT IN THEOREM 5.2, \(\epsilon_1^*\) RATHER SHOWS THE MAXIMUM \(\|\dot{w}(t)\|\) TOLERABLE BY THE ES LOOP TO MAINTAIN STABILITY AND TO SATISFY (5.21) UNDER ASSUMPTION 5.7.
5.3 Multiplexed Extremum Seeking Scheme

In the previous section it was established that ES will converge on average to $u^*(\bar{w})$ for some $\bar{w} \in \Gamma$ when the ES parameters are tuned properly. When $w(t)$ is measurable, this interpretation of the ES behavior suggests that one may benefit from this behavior by dividing $\Gamma$ into multiple smaller subregions, and estimating $u^*(\bar{w})$ in each region separately. This concept is illustrated in Figure 5.2. Suppose that $w(t) \in \Gamma \subset \mathbb{R}^1$ denotes the exogenous disturbance signal and $u^*(w)$ is the unique extremum input of the input-output mapping for any fixed $w$. In order to estimate $u^*(w)$, it is approximated in a piecewise constant fashion. To that end the domain of $w(t)$, $\Gamma$, is divided into multiple subregions, in which $u^*(w)$ is assumed constant.

The implementation is illustrated in Figure 5.3 and is referred to as “Multiplexed ES”. The multiplexed ES controller consists of independent black-box ES controllers in the adaptation function, the operation of which are governed by the activation function. The activation function plays a central role in multiplexed ES design. It is designed such that only one ES controller is activated at a time depending on which subregion $w(t)$ falls within.

Without loss of generality, it is assumed that the compact set $\Gamma$ can be divided into a finite number of subregions, each characterised by a lower-bound and an upper-bound on each dimension of of $w$, similar to the definition of $\Gamma$ in (5.2),

$$\Gamma_{i_1,i_2,\ldots,i_{N_w}} = \{ w \mid w_{i_j}^{j-1} \leq w_j < w_{i_j}^j, \text{ for } j = 1, \ldots, N_w \},$$

for some $w_{i_j}^j \in [\bar{w}_j, \bar{w}_j)$ & $i_j = 1, 2, \ldots, M_j$

where the subscript in $\Gamma_{i_1,i_2,\ldots,i_{N_w}}$ indicates the index of the subregion, and $M_j$ is the total
number of divisions in dimension $j$. Using this approach, the minimizer in each region by a single value can be approximated. Therefore $u^*(w)$ is effectively approximated in each subregion by a single value indexed as $u^*_{i_1i_2...i_{N_w}}$. This concept is shown in Figure 5.2 for the case that $N_w = 1$ (i.e. $\Gamma \subset \mathbb{R}$).

To simplify the notation, it is assumed that all the subregions are defined with equal lengths in each dimension and also across all dimensions, i.e.

$$w_i^j - w_{i-1}^j =: \Delta_w, \forall i_j = 1, \ldots M_j, j = 1, \ldots N_w.$$

Furthermore, $M$ denote the total number of subregions (i.e. $M = M_1 \times M_2 \times \ldots M_{N_w}$). Using this notation, the index of $u^*_i$ can be shortened for $i = 1, 2, \ldots M$. This strategy allows to employ $M$ independent ES controllers, one for each subregion.

The following stability analysis is focused on finding the conditions under which each ES controller, referred to as ES$_i$ for $i = 1, 2, \ldots M$, maintains its stability within this new framework. If $\bar{u}_i$ represents the current estimate of $u^*_i$ in ES$_i$, the update law for $\bar{u}_i$ and for $i = 1, 2, \ldots M$ is given as\(^3\)

\begin{align*}
\dot{x} &= f(x, u, w(t)), \\
\dot{u}_i &= -k\Omega \sigma_i(w(t))h(x, u) \sin(\Omega t_i),
\end{align*}

\(^3\)For simplicity in the notation, it is assumed that the tuning parameters $(a, k, \Omega)$ are picked the same for all extremum seekers.
\[ u = \sum_{i=1}^{M} \sigma_i(w(t))(\bar{u}_i + a \sin(\Omega t_i)), \quad (5.23c) \]
\[ \dot{t}_i = \sigma_i(w(t)), \quad (5.23d) \]

where \( \sigma_i(\cdot) : \mathbb{R}^{Nw} \to \{0, 1\} \) is the activation function output for ES\(_i\), and \( t_i \) is the amount of time when ES\(_i\) has been activated. ES\(_i\) is activated, i.e \( \dot{u}_i \neq 0 \) and \( \dot{t}_i \neq 0 \), when \( \sigma_i(w(t)) = 1 \) and is deactivated otherwise. Hence, any \( w(t) \) trajectory in \( \Gamma \) produces a sequence of activation for each ES\(_i\), formally defined as the moments \( \sigma_i(w(t)) \) goes from 0 to 1. Now let \( \{t^1_i, t^2_i, \ldots\} \) denote the activation sequence of ES\(_i\) for a given \( w(t) \) in \( t_i \) (see Figure 5.5). Hence, the activation time is given by \( t^{d_i+1}_i - t^d_i, \) for \( d_i = 1, 2, \ldots \).

The next assumption and the following remarks show how the activation function must be designed.

**Assumption 5.8.** For each \( i = 1, 2, \ldots, M \) the activation function \( \sigma_i(\cdot) \) is chosen such that it guarantees a \( \tau \in (0, \Delta_w/\epsilon_1) \) such that,
\[ t^{d_i+1}_i - t^d_i \geq \tau, \quad \forall d_i = 1, 2, \ldots \]  

(5.24)

**Remark 5.8.** According to Assumption 5.8 the activation function is designed such that they ensure a predictable minimum activation time for all ES\(_i\)'s, even in the worst case scenario. When the shape of \( w(t) \) is unknown, the worst case scenario for a given design may occur when the ex-
5.3 Multiplexed Extremum Seeking Scheme

Figure 5.6: The activation strategy is shown for $\Gamma \subset \mathbb{R}^2$. The hysteresis regions are colored grey. ES$_1$ (solid), ES$_2$ (dashed), and ES$_3$ (dash-dotted) activations are shown for an arbitrary $w(t)$.

Orogenous input chatters around the boundaries of the subregions. Therefore it is necessary to incorporate some form of hysteresis action in the activation strategy to avoid high frequency switching between neighboring ES$_i$s. First suppose that $w(t) \in \Gamma \subset \mathbb{R}$. Since it is assumed that $\|\dot{w}\| < \epsilon_1$, therefore, as shown in Figure 5.4, a hysteresis of length $\delta_w$ guarantees the minimum activation time of $\frac{\delta_w}{\epsilon_1}$. Using an arbitrary $w(t)$ signal, Figure 5.5 illustrates the activation sequence marked with different line styles.

Remark 5.9. In a multi-dimensional scenario, the activation strategy requires careful treatment. Figure 5.6 illustrates such a scenario for $\Gamma \subset \mathbb{R}^2$ for an arbitrary $w(t)$ trajectory. $\Gamma$ is divided into four subregions as shown in the figure. The activation sequence is marked by different line styles. First ES$_1$ is active. When $w(t)$ passes the hysteresis region completely at point A, ES$_3$ is activated and ES$_1$ stops working. ES$_3$ is active until $w(t)$ passes the hysteresis region completely again at point B. At point B either ES$_2$ or ES$_4$ can get activated. To solve this issue one way is to divide the hysteresis region by half and determine on which side of the division line $w(t)$ crosses the boundary. The ES to which $w(t)$ is closer gets activated, in the case of Figure 5.6 ES$_2$. Note that this strategy again guarantees a predictable minimum activation time, i.e.

$$\tau = \frac{\delta_w}{2\epsilon_1}.$$ 

This idea can be extended for higher dimensions.
The multiplexed ES controller loop in (5.23a)-(5.23d) is a hybrid system. Note that, when ES
is deactivated ($\sigma_i(w(t)) = 0$), the state of ES$_i$ is preserved but the plant states, $x$, evolve continuously in time. Therefore, at the next activation, the state of the plant as seen by ES$_i$ will exhibit a sudden change in $t_i$. This fact can be observed if the state equations (5.23a)-(5.23d) are described in $t_i$,

$$
\begin{align*}
\dot{x}(t_i) & = f(x, u, w_i(t_i)) & t_i \neq t_i^d, d_i = 1, 2, \ldots \\
\dot{x}(t_i) & \in D_i & t_i = t_i^d, d_i = 1, 2, \ldots \\
\dot{u}_i & = -k_i \Omega h(x, u) \sin(\Omega t_i), \\
u & = \bar{u}_i + a \sin(\Omega t_i),
\end{align*}
$$

(5.25a)

(5.25b)

(5.25c)
Theorem 5.3. Suppose that Assumptions 5.1-5.6 and 5.8 hold. Then for the system given by (5.27a)-(5.27c), for any \((r_x, \delta_u, \mu) \in \mathbb{R}^3 \) there exist \((a^\ast, \Omega^\ast) \in \mathbb{R}^2 \) such that for any \(a \in (0, a^\ast)\) there exist \(k^\ast, \tau^\ast\) such that for any \(\tau \in (\tau^\ast, \infty)\) there exist \((k^{**}, \Omega^{**}) \in (0, k^\ast) \times (0, \Omega^\ast)\) such that if \((k, \Omega) \in (k^{**}, k^\ast) \times (\Omega^{**}, \Omega^\ast)\), any trajectories \((\tilde{u}_i(t_i), \tilde{x}(t_i))\) originating in \(B(\delta_u) \times B^{N^\ast}(r_x)\) will satisfy
\[
\lim_{d_i \to \infty} |\tilde{u}_i(t_i^{d_i})| < \mu + \gamma(\epsilon_1), \quad \lim_{t_i \to \infty} \|\tilde{x}(t_i)\| \in D_i.
\]
for all \(i = 1, 2, \ldots, M\).

Proof: See Appendix B.4 for the sketch of proof.

The following remarks are provided to give insight about the results of Theorem 5.3.

Remark 5.10. The result of Theorem 5.3 shows that, when the multiplexed scheme is tuned properly, the sequence of \(\{\tilde{u}_i(t_1^i), \tilde{u}_i(t_2^i), \ldots\}\) converges asymptotically to a \(\mu+\gamma(\epsilon_1)\) neighborhood of the origin. However, the same conclusion can not be drawn for \(\tilde{x}(t_i)\) since it still exhibits discontinuity at \(t_i \to \infty\).

Remark 5.11. In Theorem 5.3 the minimum activation time \(\tau\) is determined based on other tuning parameters. As mentioned in Remark 5.8, when the shape of \(w(t)\) is unknown, \(\tau\) is realized through the selection of the hysteresis of length \(\delta_w\) in the activation function. On the other hand \(\delta_w\) is upper bounded by \(\Delta_w\).

Remark 5.12. Referring to Assumption 5.8, the upper bound for \(\tau\) is proportional to \(\Delta_w\) and inversely proportional to \(\epsilon_1\). Hence, it is possible that \(\tau^\ast\) is not realisable for the given \(\Delta_w\) and \(\epsilon_1\). In this case Theorem 5.3 can not be used to show stability of the scheme. In some cases this can be rectified by increasing the size of subregions, \(\Delta_w\). If no \(\Delta_w\) within the limits of \(\Gamma\) can solve the problem, a multiplexed approach is not applicable in such a case. In such a case, the stability results given in Theorems 5.1 and 5.2 for conventional (non-multiplexed) ES schemes may still apply.

Remark 5.13. In (5.26b), to obtain a finer estimation of \(u^\ast(w)\) it is desirable to reduce the size of subregions, \(\Delta_w\). However, as it can be seen from the proof in Appendix D, this may require a higher \(k^{**}\). However by increasing \(k^{**}\) the range \((k^{**}, k^\ast)\) may become empty, and as a conse-
quence no feasible solution exists. This is intuitively to be expected, as there is insufficient time spent in any one region of the multiplexed scheme.

### 5.3.1 Fast Multiplexed Extremum Seeking Scheme

The multiplexed ES scheme can be applied to a fairly general nonlinear system. However, if the plant dynamics can be approximated by a Hammerstein structure, one can take advantage of the known dynamics to use targeted ES techniques to implement the adaption function.

The purpose of the following development is to extend the multiplexed ES framework to be able to use fast black-box ES scheme [80] as shown in Figure 5.7. The next assumption is considered on the plant input–output dynamics.

**Assumption 5.9.** The plant can be represented as a nonlinear map, \( Q(\cdot) : \mathbb{R} \times \mathbb{R}^{N_w} \rightarrow \mathbb{R} \) followed by a stable LTI filter, \( F_0(s) \), which has a minimal state space representation

\[
\begin{align*}
\dot{x}_F &= A_F x_F + B_F Q(u, w(t)), \\
y &= C_F x_F + D_F Q(u, w(t)),
\end{align*}
\]

where \( x_F \in \mathbb{R}^{N_x} \) and \( w(t) \in \Gamma \subset \mathbb{R}^{N_w} \).

**Remark 5.14.** Although Assumption 5.9 reduces the class of plants from the fairly general case considered in (5.1), but it enables deployment of fast ES [80] in multiplexed framework. In addition \( Q(u, w(t)) \) holds in Assumptions 5.4 and 5.5.

The following outlines the development of the adaptation law for \( \bar{u}_i \) using fast black-box ES scheme introduced in [80].

Consider the dynamics of the \( i \)th ES block, \( ES_i \). The state equation describing \( ES_i \) internal states can be expressed as ([80]),

\[
\begin{bmatrix}
\dot{x}_i \\
\dot{\hat{u}}_i
\end{bmatrix} = \begin{bmatrix}
\Omega \sigma_i (A - LC) & \Omega \sigma_i L \\
-k\Omega \sigma_i C' & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_i \\
y
\end{bmatrix},
\]

(5.29)
5.3 Multiplexed Extremum Seeking Scheme

Figure 5.7: The schematic of multiplexed ES scheme implemented with fast ES schemes.

where

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0 \\
\end{bmatrix},
C = \begin{bmatrix}
1 \\
1 \\
0 \\
\end{bmatrix},
C' = \begin{bmatrix}
0 \\
\sin(\Omega t_i - \phi) \\
\cos(\Omega t_i - \phi) \\
\end{bmatrix}
\]

\[
u(t) = \bar{u}_i + a \sin(\Omega t_i),
\]

\[
i_i = \sigma_i(w(t)).
\]

where \(a\) and \(\Omega\) are the perturbation signal amplitude and frequency respectively, and \(k\) denotes the adaptation gain. The phase shift \(\phi\) is added in demodulation signal vector \(C'(\cdot)\) to effectively compensate for the phase shift incurred by the output dynamics \(F_o(s)\).

Finally, \(t_i\) is the time when \(\text{ES}_i\) is active, i.e. \(\sigma_i \neq 0\). For simplicity in the notations it is assumed that all \(\text{ES}_i\)'s have similar tunings. It can be shown that the observer states are

\[
\hat{x}_i = \begin{bmatrix}
\hat{Q}_0(\bar{u}, w(t)) \\
|F_o(i\Omega)|a\hat{Q}'_u(\bar{u}, w(t))(\sin(\Omega t_i + \varphi_F)) \\
|F_o(i\Omega)|a\hat{Q}'_u(\bar{u}, w(t))(\cos(\Omega t_i + \varphi_F)) \\
\end{bmatrix},
\]

where \(\varphi_F\) denotes \(\arg F_o(i\Omega)\), and \(\hat{Q}_0\) and \(\hat{Q}'_u\) respectively represent the estimates of the mean-value and the gradient of the map to be optimized.

**Assumption 5.10.** \(L\) is chosen so that \(A - LC\) is Hurwitz.
Assumption 5.11. There exist \((J, \Omega_{\text{min}}, \Omega_{\text{max}}) \in \mathbb{R}^3_+\) such that for all \(\Omega \in [\Omega_{\text{min}}, \Omega_{\text{max}}]\),
\[
\text{Re}\{\exp(i\phi(\Omega))F_o(i\Omega)\} \geq J.
\] (5.30)

Remark 5.15. The fast ES approach theoretically allows for any \(\Omega\) to be selected as long as \(\phi\) can be chosen to ensure (5.30). Unlike most traditional ES approaches, time-scale separation between the optimizer and the plant dynamics is not necessary.

The following proposition can be considered as a modification of the results in Chapter 5 where the standard black-box ES is replaced by fast black-box ES [80]. It provides the theoretical foundation for the ES development in section 8.1.

Proposition 5.1. Consider (5.28a), (5.28b), and (5.29). For any initial condition set defined by \((\beta_x, \beta_{\hat{x}}, r_u) \subset \mathbb{R}^N_x \times \mathbb{R}^3 \times \mathbb{R}_{>0}\) and frequency range \([\Omega_{\text{min}}, \Omega_{\text{max}}]\) that satisfy Assumptions 5.2, 5.4–5.6, 5.8–5.11, and for any \(\mu \in \mathbb{R}_{>0}\), there exist \(a^* \in \mathbb{R}_{>0}\) such that for any \(a \in (0, a^*)\) there exist \((\tau^*, k^*) \in \mathbb{R}_{>0}^2\) and \(\gamma(\cdot) \in \mathcal{K}\) such that for any \(\tau \in (\tau^*, \infty)\) there exist \(k^{**} \in (0, k^*)\) such that for all \(k \in (k^{**}, k^*)\), any trajectory with initial condition \((x_F, \dot{x}_i, \bar{u}_i - u^*(\bar{w}_i)) \in \beta_x \times \beta_{\hat{x}} \times (-r_u, r_u)\) will satisfy the following inequality
\[
\lim_{d_i \to \infty} |\bar{u}_i(t_{d_i}^i) - u^*(\bar{w}_i)| < \mu + \gamma(\epsilon),
\] (5.31)
for all \(i = 1, 2, \ldots, M\).

Proof. The proof of Proposition 5.1 uses the same steps as in Theorem 5.3, with the restricted class of systems in (5.28) enabling the fast black box algorithm to be used.

5.4 Simulation Example

The following simulation examples are performed to demonstrate the performance and key features of the proposed scheme. Consider a plant with the following state-space representation:
\[
\dot{x} = -x + u^2 - uw(t),
\]
\[
y = x,
\]
for some $w(t) \in \mathbb{R}$. It is a simple matter to show that the input-output map is given by $Q(u, w) = u^2 - uw$ with a minimum at

$$u^*(w) = w/2. \quad (5.32)$$

For the first simulation example, a triangular wave, as shown in Figure 5.8, is used with two different frequencies of 5 and 0.5 Hz, corresponding to $\epsilon_1$ of 0.6 and 6. Suppose that the practitioner knows that the exogenous input $w(t)$ ranges in $\Gamma = [2, 8]$ and it is measurable. For the sake of this example, the exogenous input shape will be treated as known. So it is not necessary to implement the hysteresis in the activation function to
provide a predictable minimum activation time. In this case, a minimum activation time can be guaranteed by picking $\Delta w$ appropriately.

For this simulation study $\Gamma$ is divided into three equal subregions, i.e. $M = 3$ and $\Delta w = 2$. ES$_1$, ES$_2$ and ES$_3$ are active when $w(t)$ is in $[2, 4)$, $[4, 6)$ and $[6, 8]$ respectively. All ES controllers are tuned with $a = 0.4$, $\Omega = 0.03$ Hz, and $k = 0.05$. The plant initial condition is chosen as $x(0) = 0$. The optimizer states are initialized at $\bar{u}_1(0) = -1$, $\bar{u}_2(0) = -5$, $\bar{u}_1(0) = -10$.

For space reasons, only $\bar{u}_1(t)$ is shown in Figure 5.9. It can be seen that for the higher frequency of the exogenous input, $\bar{u}_1$ has converged to a wrong value, while it has converged to the inside of the locus for the slower frequency. To find out the reason, it is instructive to look at the state of the plant as seen by ES$_1$ which is shown in Figure 5.10. It can be seen that for higher exogenous input frequencies, plant states experience more frequent and shorter discontinuities. As discussed in Appendix D, this situation causes the optimizer to start converging to a wrong value at each activation.

Next, in order to investigate the effectiveness of the multiplexed ES approach, the results will be compared with the behavior of a scheme with $M = 1$ (i.e. a regular non-multiplexed ES scheme).

As before, the plant initial condition is chosen as $x(0) = 0$. The optimizer state is initialized at $\bar{u}(0) = -1$. With $a = 0.4$, $\Omega = 0.03$ Hz, and $k = 0.05$ the performance of the ES scheme is shown in Figure 5.11 when the same triangular exogenous input

Figure 5.12: The ES output $\bar{u}(t)$ shown for a triangular exogenous input of frequency 0.5 Hz for multiplexed ES with $M = 3$ (dashed line) and $M = 1$ (grey line) against $u^*(w(t))$ (black solid line).
of frequency 0.5 Hz and 5 Hz are applied. From (5.32) and Theorem 5.1 it is expected that \( \bar{u}(t) \) converges to a close vicinity of a point \( u^*(\bar{w}) \in [1, 4] \). Specifically, since the average for the triangular wave is located at \( \bar{w} = 5 \), therefore \( \bar{u}(t) \) must converge close to \( u^*(\bar{w}) = 2.5 \).

Figure 5.12 plots \( \bar{u}(t) \) against \( u^*(t) \) for the triangular exogenous input of frequency 0.5 Hz for the multiplexed ES approach with \( M = 3 \) and \( M = 1 \). As shown in this figure, using the multiplexed approach with \( M = 3 \) results in a more accurate estimation of \( u^*(t) \).

5.5 Conclusion

First, this chapter presents a rigorous analysis that reveals the behaviour of a class of black-box ES acting on plants perturbed with exogenous disturbances. The results show that the ES output converges close to the extremum of an “average” steady-state input-output map. Second, this chapter proposes a multiplexed ES scheme as an approach for handling uncertain systems with measurable time varying exogenous disturbances. It is shown that under proper tuning, the proposed approach can inherently achieve better accuracy in terms of estimating the time varying extremum. A simulation example is provided to demonstrate the improved accuracy of the proposed method.
Part II

Implementation to Online Calibration of a CNG-fueled Engine
Chapter 6

Experimental Setup

The experimental setup used in this study is shown in Figure 6.1. A 6-cylinder Ford Falcon BF MY2006 engine, specified in Table 6.1, was connected to a low inertia Horiba-Schenck TITAN T 460 dynamometer in the Thermodynamics lab at The University of Melbourne. The layout of the hardware configuration is shown in Figure 6.2.

Table 6.1: Engine specifications

<table>
<thead>
<tr>
<th>Manufacturer</th>
<th>Ford of Australia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylinders</td>
<td>In line 6</td>
</tr>
<tr>
<td>Capacity</td>
<td>3984 cm³</td>
</tr>
<tr>
<td>Compression Ratio</td>
<td>10.3:1</td>
</tr>
<tr>
<td>Valve Train Configuration</td>
<td>DOHC Dual Independent VCT</td>
</tr>
</tbody>
</table>

The dynamometer is a 460 kW electric motor/generator with the capability to motoring the engine as well as absorbing power from the engine that generates brake force. It was controlled by a personal computer using STARS software. STARS can realize various engine operating conditions by tracking the desired torque-speed trajectories. Internal PI controllers enable the dynamometer to track the desired speed and torque trajectories by applying sufficient brake force and manipulating the engine throttle position respectively. For further details see Section 3.1 of [4].

The engine was converted to run on CNG by using a PRINS aftermarket conversion kit. The PRINS conversion kit employs six KEIHIN gas injectors that inject natural gas into the manifold just before the cylinders intake valves. As shown in Figure 6.3 the injectors deliver a linear relationship between the volume flow rate and the injectors’ opening time, which is referred to as the injection duration here. The plot was acquired experimentally and is compared against the manufacturer’s specifications. With the linear flow
rate characteristic, injection duration can be used to estimate the gas volume flow rate. In addition, the fuel mass flow measurement was available through a CMF series Coriolis flow meter.

For extremum seeking, controllers were developed in Simulink/MATLAB, and were implemented on a dSPACE MicroAutoBox. MicroAutoBox was able to control injection duration using the PRINS injector drivers. The desired spark timing was calculated by MicroAutoBox and uploaded to the production ECU via two CAN links; one between the ECU and ATI VISION software and another one between the ATI VISION software and MicroAutoBox. The reason behind having two separate CAN links was: firstly for safety reasons, since ATI VISION was able to alter sensitive engine parameters on ECU; and secondly to reduce the uncertainty in the communication delay time caused by arbitration. All other engine inputs and associated parameters were controlled by the ECU.

Brake torque was measured using a GIF torque sensor installed in the dynamometer-engine common shaft. The torque sensor measurement signal was filtered by using an
6.1 Test Fuels

The fuel was supplied from three high pressure gas bottles. The bottles were filled with pipeline natural gas using a high pressure compressor. Table 6.2 specifies the composition of a typical pipeline gas in Victoria, however the exact composition of the gas was not available. Nevertheless the composition was constant, assuming a fixed natural gas composition in between filling events. This gas is referred to as the test gas in the following chapters. Also to establish the composition dependency of optimal set-points, a premixed gas with a known composition specified in Table 6.2 was used.
Figure 6.3: Flow curve shows the linear performance of the injectors for injection durations more than 2.5 ms, and constant fuel line pressure of 3.2 bar. The manufacturer’s specification is given for a fuel line pressure of 3 bar.

Table 6.2: The composition of pipeline natural gas in Victoria [92] and the premixed gas (Gas B) in volume percent.

<table>
<thead>
<tr>
<th>Constituents</th>
<th>Pipeline gas</th>
<th>Premixed gas (Gas B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Methane</td>
<td>90.6%</td>
<td>80%</td>
</tr>
<tr>
<td>Ethane</td>
<td>5.6%</td>
<td>2%</td>
</tr>
<tr>
<td>Propane</td>
<td>0.8%</td>
<td>0.5%</td>
</tr>
<tr>
<td>Other alkane</td>
<td>0.2%</td>
<td>0%</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>1.1%</td>
<td>9%</td>
</tr>
<tr>
<td>Carbon dioxide</td>
<td>1.7%</td>
<td>8.5%</td>
</tr>
</tbody>
</table>

6.2 Test Conditions

6.2.1 Steady State Operation

In this study, the performance of the fast ES schemes developed in Chapter 3 and Chapter 4 was investigated at three representative constant operating points (i.e. constant speed and throttle position), which is relevant to stationary power plants application. The first operating point is located at 800 rpm and 2.15 % throttle position near the idle condition, the middle one is considered near the world wide mapping (WWM) point for the Ford
6.2 Test Conditions

Figure 6.4: New European Driving Cycle and the corresponding engine speed and torque for the test engine. Shaded regions correspond to idle operations.

Falcon engine at 1500 rpm and 6.25 % throttle, and the third operating point is at 2000 rpm and 13.5 % throttle.

6.2.2 Transient Operation

The NEDC specifies the vehicle speed for four repeated urban driving cycle (UDC) and one extra-urban driving cycle (EUDC). Figure 6.4 illustrates the NEDC vehicle speed trajectory, and resulting engine speed and torque trajectories recorded from a calibration test using the Ford Falcon engine.
Chapter 7
Fast Extremum Seeking for Optimization of Brake Specific Fuel Consumption

INJECTION DURATION AND SPARK TIMING are two important parameters affecting engine fuel efficiency. As such, optimal set-point values are required to guarantee fuel-efficient operation. While air-fuel ratio and spark timing are considered separately in this chapter, first it is established that the optimal set-point values are affected by fuel composition in a CNG-fueled engine. Then, the fast Extremum Seeking (ES) algorithms developed in Chapter 3 and Chapter 4 are experimentally tested to reduce the convergence time in optimizing brake-specific fuel consumption with respect to spark timing and injection duration, respectively. Finally, the performance improvements obtained by using the proposed ES schemes are compared against that of the existing ES frameworks.

7.1 Spark Timing Optimization

Previous investigations have established that the maximum brake torque (MBT) spark timing of a CNG-fueled engine varies with the gas composition [12, 27, 116]. In particular, in [84] this fact has been shown for the Ford Falcone engine using two CNG compositions. It demonstrates that the MBT spark timing values are different for the two gas. It implies that, if the engine is optimized for one gas composition, it will deliver suboptimal performance when running on the other composition. Therefore, there is a clear need to employ a means of adaptation to unknown fuel compositions.
7.1.1 Identification of Engine with Brake Torque as Output and Spark Timing as Input

To be able to use the results of Chapter 3, it is shown that the engine input-output behavior can be approximated well by a Hammerstein model, considering spark timing as input and brake torque as output. To that end, a parametrized Hammerstein model was considered, and open loop tests were performed to identify the model parameters.

In the first step, the steady-state input-output mapping was identified for the test gas by performing spark sweeps at each three operating points, where the upper and lower limits on the spark timing were found to deliver acceptable combustion. A fixed interval of 60 seconds were considered between the steps; the first 20 seconds were considered for the engine transients to settle down, and the brake torque data was averaged over the following 40 seconds.

For the WWM operating point, the mapping between the engine brake torque and spark timing is shown in Figure 7.1. In general, if \( u \) represents the spark timing in Crank Angle Degrees Before the Top Dead Centre (CAD BTDC) the brake torque in Newton-meter (Nm) can be expressed as:

\[
\tau_b = Q(u) = \theta_3 u^3 + \theta_2 u^2 + \theta_1 u + \theta_0.
\]

3rd order and 2nd order (\( \theta_3 = 0 \)) polynomials were fitted to the experimental data. For the WWM operating point, the corresponding values of \( \theta_i \)'s are presented in Table 7.1. Statistical analysis revealed that there was no significant difference between the two and therefore the 2nd order polynomial was used later for plant dynamics identification.

<table>
<thead>
<tr>
<th>Order</th>
<th>( \theta_3 )</th>
<th>( \theta_2 )</th>
<th>( \theta_1 )</th>
<th>( \theta_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3rd</td>
<td>( 6.644 \times 10^{-4} )</td>
<td>-0.0968</td>
<td>4.2633</td>
<td>25.5921</td>
</tr>
<tr>
<td>2nd</td>
<td>0</td>
<td>-0.0308</td>
<td>2.1613</td>
<td>47.2424</td>
</tr>
</tbody>
</table>

In the second step, the output dynamics of the plant were identified. The structure of the plant as seen by the controller, is shown in Figure 7.2. First, note that for a 4-stroke 6-cylinder engine the number of combustion per second is \( \frac{w_e}{20} \), where \( w_e \) is engine speed in rotation per minute (rpm). It indicates that any changes in parameters affecting the
7.1 Spark Timing Optimization

Figure 7.1: Steady-state mapping between Brake torque (Nm) and spark timing (CAD BTDC) for the test gas at WWM operating point. Smaller step size was chosen for points closer to the MBT.

Combustion properties, such as a step change in the spark timing, will be observed at the next cycle; e.g., for the engine rotating at 800 rpm it occurs within 0.025 seconds. Meanwhile, as mentioned in Chapter 6, the torque sensor output signal is low-pass filtered with a bandwidth of 2.5 Hz. Hence, the dynamics associated with the filter are significantly slower than that of the engine’s combustion-related dynamics. Subsequently it can be concluded that, in a Hammerstein approximation of the input-output dynamics, combustion-related dynamics can be ignored. The response signal from the torque sensor to two step changes in spark timing is illustrated in Figure 7.3. The unfiltered measurement data (grey line) shows how fast the engine reacts to a step change in spark timing.

As shown in Figure 7.2, there is a communication delay of $T_d$ seconds between MicroAutoBox and and ECU. This communication delay was slightly variable, and was a function of ATI VISION software workload. The slight mismatch in the left plot of Figure 7.3 between the filtered output and the identified model predicted output was due to this negligible variation.

To validate the identified dynamics, the experimentally acquired Bode diagram is plotted and compared against that of the identified model as shown in Figure 7.4. The
The parameterised model of the mapping is chosen to be represented by a quadratic function,

\[ z = Q(u) = \theta_2 u^2 + \theta_1 u + \theta_0 \]  \hspace{1cm} (7.1)

where \( \theta_i \)s are the composition-dependent coefficients, and hence are unknown. If plant is stimulated by an input of \( u = \bar{u} + a \sin(\Omega t) \), in the absence of the plant dynamics, the experimental Bode plot was obtained by applying a sinusoidal signal of fixed amplitude and variable frequency to the spark timing. Fast Fourier Transform (FFT) analysis was then used to find the amplitude gain and phase shift effect of the plant at each frequency. In fact, Figure 7.4 confirms our assumption that, the dynamics associated with the plant predominantly consists of the introduced filtering plus the time delay effect.

### 7.1.2 Controller Overview

The parameterised model of the mapping is chosen to be represented by a quadratic function,

\[ z = Q(u) = \theta_2 u^2 + \theta_1 u + \theta_0 \]  \hspace{1cm} (7.1)

where \( \theta_i \)s are the composition-dependent coefficients, and hence are unknown. If plant is stimulated by an input of \( u = \bar{u} + a \sin(\Omega t) \), in the absence of the plant dynamics, the
7.1 Spark Timing Optimization

Figure 7.4: The Experimentally acquired bode plot versus that of a 4th order Butterworth filter with a cutoff frequency of 2.5 Hz. The slight mismatch in the amplitude is due to the interference from other inputs, such as cam movements, during the experiments.

The output can be expressed as,

\[ y = \theta_2 (\ddot{u} + a \sin(\Omega t))^2 + \theta_1 (\ddot{u} + a \sin(\Omega t)) + \theta_0, \]  
\[ = \theta_2 \dddot{u}^2 + \theta_2 \frac{a^2}{2} - \theta_2 \frac{a^2}{2} \cos(2\Omega t) + 2\theta_2 \dddot{u}a \sin(\Omega t) + \theta_1 \dddot{u} + \theta_1 a \sin(\Omega t) + \theta_0, \]  
\[ = \phi_1 \left( -\frac{a^2}{2} \cos(2\Omega t) \right) + \phi_1 (a \sin(\Omega t)) + \phi_0, \]  
\[ \text{(7.2c)} \]

where \( \phi_0 = \theta_2 \dddot{u}^2 + \theta_2 \frac{a^2}{2} + \theta_1 \dddot{u} + \theta_0, \) \( \phi_1 = 2\theta_2 \dddot{u} + \theta_1, \) and \( \phi_2 = \theta_2. \) In fact, \( \phi_0 \) represents the slowly varying part of the signal for sufficiently small \( \dddot{u}, \) while \( \phi_1 \) is the gradient of the mapping \( Q(\cdot) \) at \( \dddot{u}. \) Using the representation of (7.2c), a new set of unknown parameters can be considered to be \( [\phi_2, \phi_1, \phi_0]. \) In the presence of the plant dynamics, using the notation of (7.2c), the predicted output at steady-state can then be described as,

\[ \hat{y}_G = \hat{\phi}_0 + \hat{y}_p^T \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix}, \]  
\[ \text{(7.3)} \]
where the signal vector $\hat{y}_p$ is,

$$
\begin{bmatrix}
|F_o(i\Omega)| a \sin (\Omega t + \arg(F_o(i\Omega)) - T_d\Omega) \\
-|F_o(i2\Omega)| \frac{a^2}{2} \cos (2\Omega t + \arg(F_o(i2\Omega)) - 2T_d\Omega)
\end{bmatrix}.
$$

Finally the state equation for the parameter estimator is given as,

$$
\begin{bmatrix}
\dot{\hat{\phi}}_0 \\
\dot{\hat{\phi}}_1 \\
\dot{\hat{\phi}}_2
\end{bmatrix} = \Omega \begin{bmatrix}
1 \\
|F_o(i\Omega)| a \sin (\Omega t + \arg(F_o(i\Omega)) - T_d\Omega) \\
-|F_o(i2\Omega)| \frac{a^2}{2} \cos (2\Omega t + \arg(F_o(i2\Omega)) - 2T_d\Omega)
\end{bmatrix} (y - \hat{y}_G).
$$

For comparison, in conventional model-based approach that is based on time-scale separation tuning, the dynamics can be ignored due to the fact that $F_o(i\Omega) \approx F_o(i2\Omega) \approx 1$ and $T_d\Omega \approx 0$ at low frequencies. Hence, the parameter estimator is simplified as,

$$
\begin{bmatrix}
\dot{\hat{\phi}}_0 \\
\dot{\hat{\phi}}_1 \\
\dot{\hat{\phi}}_2
\end{bmatrix} = \Omega \begin{bmatrix}
1 \\
|F_o(i\Omega)| a \sin (\Omega t) \\
-\frac{a^2}{2} \cos (2\Omega t)
\end{bmatrix} (y - \hat{y}_{ss}),
$$

where $\hat{y}_{ss}$ denotes the estimation of the steady state plant output which is equal to,

$$
\hat{y}_{ss} = \hat{\phi}_2 \left(-\frac{a^2}{2} \cos (\Omega t)\right) + \hat{\phi}_1 \left(a \sin (\Omega t)\right) + \hat{\phi}_0.
$$

**Remark 7.1.** For implementations, the off-the-shelf version of the fast model-based algorithms presented in Section 3.1.4 was slightly modified to consider the fact that the output dynamics were known and fixed across all operating points. Therefore, the knowledge of the output dynamics was used to construct the parameter estimator.

A gradient ascent optimization algorithm was used for extremum seeking. To implement the optimizer, the estimation of the gradient of the mapping is required, which for the estimated 2nd order map is

$$
Q'(\bar{u}) = 2\hat{\theta}_2 \bar{u} + \hat{\theta}_1 = \hat{\phi}_1,
$$

where $\hat{\theta}_1$, $\hat{\theta}_2$ are the estimation of $\theta_1$, $\theta_2$. Therefore the adaptation law for $\bar{u}$ was implemented as,

$$
\dot{\bar{u}} = k\Omega \hat{\phi}_1
$$
7.1 Spark Timing Optimization

![Controller schematic]

Figure 7.5: Controller schematic.

7.1.3 Experimental Results

The experimental performance of the fast model-based ES scheme to locate MBT spark timing is explored and compared against that of the conventional model-based framework ([89]). The tuning parameters include: the dither signal amplitude and frequency, the adaptation gain, the initial states of the parameter estimator (7.5) and optimizer (7.9). In this chapter, the effect of dither frequency on stability and convergence properties of the model-based framework and fast model-based scheme is of primary interest. Therefore the tuning effort was focused on this parameter, while identical tuning were used for the remaining parameters.

The adaptation gain $k$ was chosen to be 0.25 to provide sufficient time-scale separation between the estimator and the optimizer. The dither amplitude tuning was a trade off between the convergence speed and accuracy of ES, i.e. a higher dither amplitude was desirable to achieve a higher convergence rate, but it caused a larger oscillation around the converged value. A dither amplitude of 1.5 CAD BTDC was chosen. $\hat{\phi}_0$ was warm started to at 80 Nm, while other parameter estimator states were initialized at zero. For dither frequency, an upper bound of 4 Hz was effectively dictated by the engine speed and the communication rate between MicroAutoBox and ECU.

Figures 7.6- 7.8 show the experimental results for the WWM operating point. Initialized at 22 CAD BTDC, ES should be able to regulate the spark timing close to the MBT value, which for the WWM operating point is located at 35 CAD BTDC. While the fast model-based ES controller can employ dither frequencies as high as 1.9 Hz, the model-based framework exhibited unstable performance at frequencies higher than 0.7 Hz. It is also noted that the accuracy of the model-based ES deteriorated for higher frequencies,
Figure 7.6: Model-based ES performance for a dither frequency of 0.4 Hz, dither amplitude of 1.5 CAD BTDC, and optimizer gain of 0.25.

Figure 7.7: Fast model-based ES performance for a dither frequency of 1.7 Hz, dither amplitude of 1.5 CAD BTDC, and optimiser gain of 0.25.

whereas it was not affected in the fast model-based ES. By employing a faster dither frequency, a shorter convergence time was achieved. The convergence time was decreased with a factor of 4.

The parameter estimator algorithm delivers a local estimate of the unknown map, which can be different from the “global” fit shown in Figure 7.1. This is an inherent limitation of the estimation algorithm. Nevertheless, since the optimizer makes use of the estimated local gradient, \( \hat{\phi}_1 \), it is not crucial to estimate the parameters globally. As shown in Figures 7.6-7.8, the estimated gradient converges and oscillates around zero. The oscillation in the estimated gradient was caused by cyclic variations in the combustion, plus an additive noise of unknown origin that was present during the experiments. The noise issue was magnified at higher frequencies, since at higher frequencies ES becomes more sensitive to any noise. Not predicted by the theorem, this phenomenon made increasing the dither frequency beyond 1.9 Hz impractical in fast model-based ES.
7.2 Injection Duration

Brake Specific Fuel Consumption (BSFC) is defined as the mass flow rate of fuel per unit output power, and is calculated as,

$$\text{BSFC} := \frac{\dot{m}_f}{\tau_b \times w_e},$$

where $\dot{m}_f$ denotes the mass flow rate of the gas, $\tau_b$ is the brake torque measured at the crank shaft and $w_e$ is the engine speed. Here, the injection duration is taken as the control (optimizing) input. Since at a given speed and throttle position mass air flow is constant, therefore by manipulating injection duration the air-fuel ratio is effectively varied.

Fuel composition can affect the optimal injection duration set-point with respect to BSFC. This fact is shown in Figure 7.9 for two different gas compositions as specified in Table 6.2, at WWM operating point. At this operating point, the minimum BSFC occurs at 5.7 ms/cycle for the test gas and 7 ms/cycle for the premixed gas composition B. It indicates that conventional static calibration for one gas composition will result in sub-optimal performance when running on the other composition. Hence, online recalibration is needed to account for the composition variation of the fuel.

7.2.1 Identification of Engine with Brake Torque as Output and Injection Duration as Input

Since the response time of the Coriolis fuel flow meter was not sufficiently fast to provide $\dot{m}_f$ for calculation of BSFC for ES, the injectors’ calibration map, shown in Figure 6.3, was
Fast Extremum Seeking for Optimization of Brake Specific Fuel Consumption

Figure 7.9: Steady-state mapping between BSFC and injection durations for two gas compositions at the operating point of 1500 rpm and 6.25 % throttle position. For the test gas, the optimum injection duration occurs around 5.8 (ms/cycle), whereas it is around 7 (ms/cycle) for Gas B specified in Table 6.2.

Figure 7.10: Engine as a plant with the injection duration as input and BSFC as output.

used to calculate a volume-based BSFC defined as,

\[
\text{BSFC} := \frac{\dot{V}_f}{\tau_b \times w_c}.
\]  

(7.11)

In fact, when the fuel line pressure and temperature are kept constant the gas volume flow rate (calculated from injection duration) is proportional to the gas mass flow rate. Hence, in injection duration calibration, the plant input was considered to be the injection duration, and the performance metric (output) was considered to be the volume-based BSFC as shown in Figure 7.10.

During a separate set of experiments conducted for the three operating points, first it was established that the engine input-output behavior was well approximated by a Ham-
merstein structure, considering the injection duration $u$ as input and the brake torque $\tau_b$ as output (see Figure 7.10). To that end, the static mapping was identified for the test gas using an “injection sweep test”, where injection duration was varied in steps of 0.1 (ms/cycle) and intervals of 60 seconds. The steady state mapping between injection duration and brake torque was obtained by averaging the output over the last 40 seconds. A quadratic polynomial provided a good fit to the mapping for the two composition. For the test gas (specified in Table 6.2) it was obtained as,

$$f(u) = -12.6534u^2 + 165.6649u - 454.3996,$$

where $u$ is the injection duration in milliseconds per cycle. Then, a randomly generated sequence was applied to injection duration and the output dynamics were identified using System Identification toolbox in MATLAB. It was found that a fourth-order bi-proper linear dynamics was a good approximation to the engine input-output characteristics as:

$$F_o(s) = \frac{0.04533s^4 + 2.364s^3 + 237s^2 + 3805s + 1.606e05}{s^4 + 28.18s^3 + 1306s^2 + 1.905e04s + 1.606e05}$$

Alternatively, as described in Section 7.1.1, it could simply be assumed that the output dynamics were consisted of those of the low-pass filtering, while fast combustion-related dynamics were ignored. For comparison, the experimentally acquired bode diagram is plotted against that of the identified $F_o(s)$ and the torque sensor low-pass filter in Figure 7.11. It shows that for frequencies of up to 2 Hz, both approaches give reasonable approximation.

To calculate the gas volume flow rate in equation 7.11, the injector calibration map was used. As illustrated in Figure 6.3, the gas volume flow rate can be considered proportional to the injection duration. The injector dynamics are fast compared to the brake torque dynamics, i.e. it can be assumed that $H_o(s) = 1$. Hence the engine as a plant with injection duration as input and BSFC as output can represented as shown in Figure 7.10. Hence, the plant’s structure falls into the class of systems considered in Chapter 4 (see Figure 4.1).
7.2.2 Controller Overview

The fast ES controller developed in Chapter 4 was used to reduce the convergence time of ES in the calibration of injection duration. The ES control structure is shown in Figure 7.12. As described in Section 4.1, the compensator \( \hat{F}_o(s) \) must satisfy Assumption 4.2. The compensator design requires some knowledge about the engine torque dynamics at frequency \( \Omega \). Since in our application the engine output dynamics \( F_o(s) \) were fixed across all operating points and all compositions, therefore it was treated as completely known. Therefore, the compensator was implemented as \( \hat{F}_o(s) = F_o(s) \). Moreover, the Luenberger observer was designed to Assumption 2.7, by the choice of \( L = [0.237, 0.343, 0.167] \). Finally, the phase shift \( \phi(\Omega) \) was tuned to \( \text{arg}(F_o(i\Omega)) \) to satisfy Assumption 2.8 in Chapter 4.

7.2.3 Experimental Results

The performance of the fast ES developed in Chapter 4 was investigated to optimize BSFC as the performance measure. The results shown in this chapter only include the WWM.
operating point at 1500 (rpm) and 6.25% throttle position. All tests were carried out with the test gas. The ES was initialized from $\bar{u}(t_0) = 6.6$ (ms/cycle), the farthest point from the optimum point at $u^* \approx 5.7$ (ms/cycle) with an acceptable combustion, yet this value of $\bar{u}(t_0)$ is still representative of the optimal $u$ using a different gas composition. Since the effect of dither frequency on the performance of the applied schemes was of primary interest, little effort was made to tune the other parameters. The dither frequency was upper-bounded by the engine speed, since the injection duration was set once per cycle.

The dither signal amplitude was set at 0.1 (ms/cycle). The first observer state $\hat{x}_0$, which represents the current BSFC, was initialized by using the calibration information. This is a reasonable approach since it can also be taken from the BSFC measurement at sample before enabling the ES. Alternatively it could also be directly set by running the observer before enabling the optimizer. This allows all observer states to settle down before the optimizer starts.

The experimental results are given in Figures 7.13–7.16. The optimizer output overlaid on the engine injection duration input and volume based BSFC are plotted for each run. For comparison, the performance of a black-box ES ([111]) was investigated first, which is shown in Figure 7.13. To achieve time-scale separation, a dither frequency of 0.25 Hz was used, and $k$ was tuned to 0.4. The best settling time achievable was around 50 seconds. By increasing $k$ further, $\bar{u}(t)$ as $t \to \infty$ departed more from $u^*$, as predicted by the results in [111].
Figure 7.13: Black-box ES [111] performance for the dither frequency of 0.25 Hz and optimizer gain of 0.4.

Figure 7.14: Fast ES scheme performance for the dither frequency of 1 Hz and optimizer gain of 0.4.

Figure 7.15: Fast ES scheme performance for the dither frequency of 1 Hz and optimizer gain of 1. Increasing the optimizer gain led to an undesirable performance.

Figure 7.16: Fast ES scheme performance for the dither frequency of 2 Hz and optimizer gain of 0.4. Increasing the dither frequency led to an undesirable performance.

Figures 7.14–7.16 present the performance of the fast ES scheme. To show the repeatability of the results, two consecutive runs are displayed. The thick dashed line indicates the moment of enabling the ES. In Figure 7.14, it can be seen that for a dither frequency of 1 Hz and an optimizer gain of 0.4 a convergence time of about 10 seconds was achieved, which is an order of magnitude faster the conventional ES scheme. The results here show that for this operating point the BSFC was improved by 8% in 10 seconds.

Further improvement in convergence for this application might be considered through either increasing the optimizer gain or dither frequency. However, as shown in Figure 7.15 and 7.16, further increases in these parameters led to undesirable closed loop be-
haviors. Nevertheless, the order of magnitude improvement relative to the conventional scheme represents a significant transient benefit.

### 7.3 Conclusion

Set-point calibration of spark timing and injection duration was considered for a CNG-fueled engine at a constant operating point. It was established that CNG composition affects the optimal set-points for injection duration and spark timing. Using system identification, it was shown that engine input–output behavior can be modeled as a Hammerstein plant from the spark timing to the brake torque measurement. Then, the fast model-based ES of Chapter 3 was demonstrated to reduce the ES convergence time in spark timing calibration by a factor of 4, compared with a model-based ES.

For injection duration, first it was shown that engine input–output dynamics can be modeled as a Hammerstein-Wiener plant from injection duration to BSFC measurement. Then, the modified fast black-box ES developed in Chapter 4 was implemented and compared against the black-box ES framework. The fast black-box ES exhibited an order of magnitude improvement in the convergence time of injection duration. In addition, the experimental results in this chapter demonstrated that the bandwidth of the actuator and the presence of the noise in the measurement may limit the range of frequencies that can be used in a fast ES.
Engine optimal set-points depend on engine speed and load, normally defining the engine operating point. In a typical driving scenario the operating point varies constantly, requiring fast adjustments of engine inputs. Traditional ES implementations that target steady-state optimization are too slow to be deployed considering the transient condition. Conventionally, a feedforward approach is taken to adjust the the engine inputs quickly. In this quasi-steady state approach, optimal engine set-points are calibrated during steady-state operation and stored in lookup table formats for a finite number of engine operating points. In the lookup tables, the optimal set-points are parametrized against the engine speed and load values. Engine inputs are then determined by employing interpolation methods for all possible operating points [37]. In the case of alternative fuels, however, this approach delivers suboptimal performance when the fuel composition changes from the one used for calibration.

To address this gap, in this chapter the fast multiplexed extremum seeker developed in Chapter 5 is proposed to optimize spark timing during a transient driving cycle. This approach is depicted in Figure 8.1. It enables the lookup table entries of the feedforward function to adapt as the fuel composition changes, thus it can be considered as an adaptive feedforward algorithm. The multiplexed ES scheme is experimentally tested for a CNG-fueled engine with unknown fuel composition over the New European Driving Cycle.

---

A substantial part of this chapter has been submitted as [104].

103
Cycle. The experimental results show that under proper tuning, the proposed controller can improve fuel efficiency for unknown natural gas compositions without requiring gas composition sensing at little additional calibration effort.

8.1 Experimental Results

The fast multiplexed ES scheme was experimentally tested to optimize fuel efficiency with respect to the spark timing for a CNG-fueled engine with unknown fuel composition over the New European Driving Cycle (NEDC). Engine was considered as a plant with the spark timing as the input and engine brake torque as the performance metric. At all times except the idle operation, ES controller optimized brake torque with respect to the spark timing to find the so-called Maximum Brake Torque (MBT) spark timing. Previous studies [41,84] have established the variation of MBT spark timing with fuel composition.

Since at all times Air-Fuel Ratio (AFR) was kept nominally constant, the relative improvement in brake torque was considered to be equivalent to that of the brake specific fuel consumption. A Heated Exhaust Gas Oxygen (HEGO) sensor was employed by the ECU to detect the relative AFR and keep the combustion at stoichiometric AFR. Assuming the constant stoichiometric AFR allowed to correspond a lower amount of air inducted into the cylinders with a lower fuel consumption at any given constant engine speed and brake torque (i.e. brake power). However, during the transient experiments,
it was found that ECU can not keep stoichiometric AFR at all times for the first time, but it improved over time due to the adaptive tables inside ECU to cope with the effect of unknown fuel composition on AFR control. Alternatively a wide-band UEGO sensor can be used with a closed-loop controller to speed up the adjustments needed to keep the AFR at stoichiometric for any unknown fuel composition, although this was not pursued in the experiments.

In the first step, a standard calibration procedure was performed to allow benchmarking of the ES controller. The MBT spark timings were obtained for a finite number of operating points using spark-sweeps. The operating points were chosen by a 30-point grid of \{(0.15, 0.2, 0.25, 0.3, 0.4, 0.6) \times \{800, 1100, 1250, 1350, 1500\}\} in load-speed plane, where engine load is defined as the fraction of the amount of air inducted into the cylinders to its theoretical maximum value.

The calibration result is illustrated in Figure 8.2. At each operating point the lower bound (the most advanced) and upper (the most retard) spark timing were found experimentally to avoid knock and high exhaust metal temperature respectively. Other engine inputs were kept constant and the AFR was regulated at the stoichiometric ratio at all times. The intake and exhaust cam were set at \(-10\) degrees with \(0\) degrees overlap. Since the cams can not be controlled precisely at the desired angles (normally with \(2\) degrees accuracy), the set points were selected in order to achieve the least sensitivity in the manifold absolute pressure (MAP) with respect to the cam movements.

The experimental results are presented in three parts. First, the controller development steps are presented. Before testing the controller over a driving cycle, it is tested in a steady-state operating condition, where engine speed and torque are kept constant. The performance of the controller in this case gives insight about the achievable improvement in a driving scenario. Finally the controller is tested over the NEDC.

### 8.1.1 Controller Development

In order to apply multiplexed ES, it is required to satisfy the relevant assumptions leading to Proposition 5.1.
Validity of Assumption 5.9

In general the engine brake torque \( \tau_b \) (Nm) can be described as a function of spark timing \( u \) (degrees BTDC), engine speed \( w_e \) (rpm), and engine load \( L \) (fraction), when other engine inputs are kept constant, i.e.,

\[
\tau_b = Q(u, w_e, L).
\]

Moreover, in Section 7.1.1 it was established that the engine as plant with spark timing as input and the brake torque as output can be represented as a Hammerstein plant. Therefore Assumption 5.9 is satisfied.

Validity of Assumptions 5.2 and 5.4

It is well established that MBT spark timing, i.e. the value of \( u \) that maximizes \( Q(\cdot) \), can be described as a function of engine speed and load \( u^*(w_e, L) \) [43]. Hence, in an online calibration scenario, engine speed \( w_e(t) \) and load \( L(t) \) are regarded as exogenous disturbances, the measurements of which are readily available. In addition it can be shown that the rate of change of both quantities are bounded, mainly due to the inertia associated with the engine rotational mechanics for \( w_e(t) \), and the throttle and manifold dynamics for \( L(t) \). Thus the conditions of Assumptions 5.2 and 5.4 are met.

Validity of Assumption 5.6

In practice the orthogonality assumption requires some knowledge about the driving scenario. In general, the variation in \( L(t) \) and \( w_e(t) \) should not have a \( \Omega \) and \( 2\Omega \) frequency components. It is implausible in practice that one drives in such a way that the variations in \( w_e(t) \) and \( L(t) \) violate (5.14a)-(5.14b).

Validity of Assumption 5.8

The activation function plays a central role in the implementation of multiplexed ES. Since the driving scenario is typically unknown and hence the engine speed and load
8.1 Experimental Results

Figure 8.2: MBT spark timing for the test gas. The operating point of 1380 rpm and 0.29 load is marked with a star.

variations are unknown, to satisfy Assumption 5.8, the activation function must be designed to guarantee a minimum activation time. The proposed activation scheme is illustrated in Figure 8.4 for an arbitrary \((w_e(t), L(t))\) trajectory. It incorporates hysteresis regions to avoid the high frequency activation/deactivations of neighboring subregions, if the engine load and speed chatter around the boundaries for a long period of time. From ES\(_1\), when \(w(t)\) passes the hysteresis region completely at point A, ES\(_2\) is activated and ES\(_1\) stops. ES\(_2\) is active until \(w(t)\) passes the hysteresis region completely again at point B. At point B either ES\(_6\) or ES\(_7\) may be activated. To solve this issue one way is to divide the hysteresis region by half and determine on which side of the division line \((w_e(t), L(t))\) crosses the boundary. The ES to which \((w_e(t), L(t))\) is closer gets activated, in the case of Figure 8.4 it is ES\(_6\).

It was found that a combination of “dead time” and hysteresis action can deliver a superior performance. The dead time is to be implemented to activate the optimizer in the ES controllers, \(t_d\) seconds after the activation moment. Essentially sufficiently long dead time avoids the transient behavior of the observer states to affect the ES performance. Particularly at high acceleration and decelerations, where the hysteresis action alone can not provide sufficiently long activation time, a short activation time can make the ES con-
controllers unstable. Alternatively, a lower optimizer gain could potentially be used which results in an undesirable slow convergence rate.

A hysteresis length of $\delta_{we} = 50$ rpm and $\delta_L = 0.01\%$ were identified to avoid high activation/deactivation rates. A dead time of $t_d = 1.5$ seconds were found experimentally to improve the performance of the ES scheme compared with the cases where $t_d = 0$.

Validity of Assumption 5.10

Assumption 5.10 can be satisfied by the choice of $L^T = [1, 1, 1]$ to produce a stable observer.

Validity of Assumption 5.11

To satisfy Assumption 5.11, the phase diagram in Bode plot of Figure 7.4 was used. In other words, $F_o(s)$ was considered known and $\phi(\Omega)$ was tuned equal to $\arg (F_o(i\Omega))$.

8.1.2 Constant Torque-Speed Test

A preliminary test was conducted at a constant speed of 1380 rpm and a constant torque of 65 Nm to demonstrate the possible improvement that can be achieved by employing
8.1 Experimental Results

ES. To maintain a constant torque, the dynamometer manipulates the throttle in the same way as a human driver. For the spark timing of 22 degrees BTDC, the corresponding load was achieved at 0.29%, thus it activates ES\textsubscript{18}. The MBT spark timing is at around 35 CAD BTDC for the engine load between 0.27% to 0.29% as shown in Figure 8.2.

The performance of the ES controller is shown in Figure 8.5 for the tuning described in Table 8.1. In the plots of Fig 8.5, spark timing is shown with engine throttle and load. Starting at a non-MBT spark timing, ES will effectively perturb the spark timing to estimate the gradient and shift it towards the MBT for the current load and speed [80]. Shifting spark timing to MBT will result in a slight increase in the brake torque which will eventually be compensated by the dynamometer torque controller through manipulating the throttle position. Although both the ES controller and torque controller can affect the brake torque, ES controller is able to distinguish between these two essentially by using a distinct dither frequency. Therefore the combined action of the ES controller and the dynamometer torque controller brought the engine to a new operating point,

Figure 8.5: Spark timing ES at a constant torque-speed mode.
where the same engine speed and torque is realized with 0.02 less load, and hence improved fuel efficiency. It is noted that similar results using a single ES controller have been demonstrated in [41, 84].

### 8.1.3 Driving Cycle Results

The distribution of the operating points in the speed-load plane during one NEDC is shown in Figure 8.6, where the subregion indexes are shown in each subregion. The operating points below 800 rpm and 0.15 bar, that are associated with the idling events, are not shown. At the idling events the MBT spark timing is not applied, due to the fact that spark timing is employed by a different control loop to maintain the idle speed [45]. Therefore no adaptation is applied in those operating point regions.

In Figure 8.6, the higher concentration of points represents higher activation duration in one NEDC. It can be seen that some of the operating subregions were not visited or only visited for short durations.

One complete NEDC plus four extra UDC was considered for the experiment to be...
able to have convergence in multiple subregions. Figure 8.7 shows the activation signals for the subregions that were activated frequently during our experiments. Only subregions 6, 12, 18, and 25 had sufficiently long activation times. The tuning used for the corresponding ES controllers are presented in Table 8.1. Different tuning values were resulted from different accelerations, \((\dot{w}_e(t), \dot{L}(t))\), experienced during the driving cycle.

The experiment results are shown in Figure 8.8 for the tuning that is presented in Table 8.1. In the figure the response for \(\bar{u}_{6}, \bar{u}_{12}, \bar{u}_{18}, \text{ and } \bar{u}_{25}\) are shown against their respective time axes. All \(\bar{u}_i\)s were started from 25 degrees BTDC. This choice of initial condition was made to provide a significant difference from the actual MBT values to better illustrate the convergence properties.

**Figure 8.7:** Activation signals are shown for the subregions during one NEDC and four extra UDC.

**Table 8.1:** ES tuning parameters

<table>
<thead>
<tr>
<th>Index (i)</th>
<th>(\Omega_i) (Hz)</th>
<th>(\phi_i) (deg.)</th>
<th>(k_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.7</td>
<td>-100</td>
<td>0.1</td>
</tr>
<tr>
<td>18,6</td>
<td>0.5</td>
<td>-60</td>
<td>0.1</td>
</tr>
<tr>
<td>25</td>
<td>0.7</td>
<td>-100</td>
<td>0.05</td>
</tr>
</tbody>
</table>

In Figure 8.8 the dashed line shows the interpolated MBT spark timing using the calibration map shown in Figure 8.2 at the current speed-load condition. It can be seen
that, although the MBT was constantly variable, the multiplexed ES has converged to a value which on average is very close to MBT.

Note that the results here were achieved without using any explicit knowledge about the driving cycle and the signals $w_r(t)$ and $L(t)$. The downside of using multiplexed approach is the piecewise-constant assumption on the MBT spark timing in each subregion. For example, the value of MBT spark timing in region 6 shows a 10 degrees variation. However, it is always possible to define finer subregions so that the assumption of piecewise-constant MBT is more realistic. By defining finer subregions the activation times become shorter and the convergence can be very challenging. Hence, there is a trade off between the size of the subregions and the stability of the approach.

### 8.2 Conclusion

In this chapter, the fast multiplexed ES of Chapter 5 was experimentally tested in spark timing calibration of a CNG-fueled engine over the NEDC. The approach was motivated by the shortcomings of existing extremum seekers to estimate MBT spark timing for engines operating in transient driving conditions. Assuming that MBT spark timing is constant in different sufficiently small speed-load subregions, multiplexed ES was demonstrated to provide a good estimation of the MBT spark timing in each subregion. This was achieved despite the transient nature of the driving cycle, and independent of using any explicit knowledge of the driving cycle in the design of the controller.
Figure 8.8: The output of the ES, s are plotted against their respective time axes $t_i$. The dashed line shows the interpolated MBT value from the lookup table for the test gas, and the black line marks the end of the first NEDC round.
Chapter 9
Contributions and Future Work

9.1 Contributions to Extremum Seeking Theory

9.1.1 Fast Model-based Extremum Seeking on Hammerstein Plants

A “fast model-based” ES scheme was developed in Chapter 3 for Hammerstein plants with a known input-output mapping parameterized by some unknown parameters. It was shown that, when the relative degree of the output dynamics is known, a filtering stage and a special parameter estimator can be designed to allow any sufficiently high dither frequencies to be used. Hence, the proposed ES can benefit from warm starting the estimator, as well as an accelerated performance achieved by using a high dither frequency. Since the optimizer and parameter estimator adaptation rates are scaled with the dither frequency, that is equivalent to achieving an arbitrary high convergence rate.

9.1.2 Fast Black-box Extremum Seeking on a Class of Hammerstein–Wiener Plants

The fast black-box ES introduced in [79, 80] consider the plant input-output dynamics to be represented by a Hammerstein or a Wiener–Hammerstein model. This requirement was modified in Chapter 4, to extend the fast black-box ES results to a class of Hammerstein–Wiener plants where the metric is a nonlinear function of two system outputs each represented by a Hammerstein structure. It was shown how a compensator stage must be designed on the plant outputs such that the input–metric relationship complies with the fast black-box ES requirements. It is worth mentioning that, this structure was considered to include the engine input-output behavior with injection duration as input and Brake Specific Fuel Consumption (BSFC) as output.
9.1.3 Analysis of Black-box ES in the Presence of Disturbance

The standing assumption in most ES implementations in the literature is that, the plant exhibits a stationary input-output behavior. However, this is not applicable to the large class of plants with a time varying input-output behavior caused by the presence of exogenous disturbances. Therefore, in Chapter 5 the application of ES acting on dynamical plants subject to slowly varying exogenous disturbances was considered. A rigorous analysis was presented to analyze the effect of a slowly varying exogenous disturbance on the behavior of a widely used black-box ES introduced in [111]. The main result established that, when $\Gamma$ denotes the range of the disturbance signal $w(t)$ and $u^*(w)$ denotes the mapping between the disturbance and the extremum point, under proper tuning the black-box ES output converges to a close neighborhood of $u^*(\bar{w})$ for a $\bar{w} \in \Gamma$. $\bar{w}$ can be considered as an average for $w(t)$.

9.1.4 Multiplexed ES for Systems with Time-varying Extremum Caused by a Measurable Disturbance

Following the previous contribution, a novel ES technique called “multiplexed” ES was developed in Chapter 5 that can be used for plants disturbed by a measurable slowly varying disturbance. Strictly speaking, when the plant is subject to a measurable disturbance $w(t) \in \Gamma \subset \mathbb{R}^{N_w}$, and $u^*(w)$ denotes the extremum input as a function of $w(t)$, multiplexed ES estimates $u^*(w)$ across $\Gamma$. Multiplexed ES is achieved by dividing $\Gamma$ into multiple subregions where $u^*(w)$ is assumed constant. This piecewise-constant assumption is the essence of multiplexed ES. Therefore, the multiplexed ES objective is to estimate $u^*(w)$ in different subregions of $\Gamma$. Hence, it can also be regarded as an adaptive feedforward controller, where the feedforward block consists of a lookup table whose entries are the estimation of $u^*(w)$ in different subregions of $\Gamma$. It was shown that, multiplexed ES can achieve better accuracy than conventional ES implementations, despite the time-varying $w(t)$. In addition, it was shown that when the plant can be approximated by a Hammerstein structure, a fast multiplexed ES can be implemented to achieve accelerated performance.
9.2 Contributions to Online Calibration of CNG-fueled Engines

ES as an online optimization method was proposed to account for the composition variation of the fuel. In the second part of the thesis, first the ES schemes developed in Chapter 3 and 4 were experimentally tested in the calibration of spark timing and injection duration with respect to Brake Specific Fuel Consumption (BSFC) for a CNG-fueled engine. Then, the novel multiplexed ES algorithm developed in Chapter 5 was utilized for spark timing calibration over a transient driving cycle. The main contributions of the second part are as follows.

9.2.1 Experimental Demonstration of Fast Model-based ES in Set-point Calibration of Spark Timing

First the engine was considered as a plant, with spark timing as input and brake torque measurement as output. Brake torque measurement was provided by a torque sensor and was filtered to remove the high frequency noise. While all other engine inputs were kept constant, it was shown that the engine input–output dynamics can be approximated well by a Hammerstein structure. In the Hammerstein model, the input–output mappings was proved to be a function of the fuel composition, while the plant dynamics dominated by those of the low-pass filtering applied on the brake torque measurement.

Then for comparison, the fast model-based ES and a model-based ES of [89] were implemented. For the first time, it was experimentally demonstrated that by using the fast model-based ES the convergence time was reduced by an order of magnitude compared with the conventional model-based ES. Further, while the fast model-based ES can theoretically achieve an arbitrary high convergence rate, the experimental investigation proved the presence of noise and actuator bandwidth were the limiting factors.

9.2.2 Experimental Demonstration of Fast ES in Set-point Calibration of Injection Duration

Considering the injection duration as the optimizing input, first a system identification process was conducted. A Hammerstein structure was shown to be a good approximation of injection duration–torque measurement dynamics, where the mapping is composition dependent and the output dynamics are associated with those of the low-pass filter
applied on the measurement. Also, the injectors’ calibration maps were used to calculate
the volume rate of fuel injected per injection duration, while their dynamics were ig-
nored due to their almost instantaneous response. As a result, the Hammerstein-Wiener
structure of Chapter 4 was established as a good approximation of engine dynamics from
injection duration to BSFC.

For comparison, both the fast ES introduced in Chapter 4 and the conventional black-
box ES of [111] were implemented. The experimental results demonstrated an order of
magnitude improvement in the convergence time, when the known knowledge of the
plant dynamics was used. Moreover, while the main result of Chapter 4 allows any suf-
ficiently high frequencies to be used, it was effectively upper-bounded by the engine
speed, as the injection duration can change once per combustion cycle.

9.2.3 Experimental Demonstration of Multiplexed ES in Optimization of Spark Tim-
ing over a Driving Cycle

For the first time, ES was implemented in spark timing calibration over a transient driving
cycle. The New European Driving Cycle (NEDC) was considered as the representative
of a typical driving scenario. The fast multiplexed ES developed in Chapter 5 was used
to adapt the MBT spark timing to the unknown fuel composition over the NEDC. To
implement multiplexed ES, first the speed-load plane was divided into 25 subregions,
inside which the MBT spark timing was assumed constant. Then, starting from any initial
spark timing that may represent the MBT spark timing for a different fuel composition,
the multiplexed ES controller estimated the MBT spark timing in each subregion, during
the operation of engine in the associated subregion. In addition, an activation function
strategy was designed to ensure a minimum adaptation period.

Since the engine dynamics from spark timing to brake torque measurement was iden-
tified as a Hammerstein plant in Chapter 7, the fast black-box ES of [80] was used for
the implementation of multiplexed ES. The experimental results discussed in Chapter 8,
showed that despite the transient nature of NEDC, the multiplexed ES was able to adapt
the MBT spark timing to the unknown fuel composition at a number of engine speed-load
regions that were visited long enough.
Future research directions are proposed to supplement the findings of this research.

9.3.1 Extension to Consider Input Constraints in fast ES

Recently, several studies have considered different methods to deal with input constraints in ES. To handle input-constraints when input is 1 dimensional, a projection-based algorithm is proposed in [76], while anti-windup and penalty-function methods are considered in [110]. Both approaches consider static optimization cases. Therefore, a further analysis to extend the result to dynamical plant is of interest. This is particularly crucial for fast ES, since the accelerated estimator tuning (using high dither frequencies) may increase the chance of violating the input limits.

9.3.2 Further Extension of Fast Model-based ES to Use the Whole Range of Dither Frequencies

In fast model-based ES developed in Chapter 3 only sufficiently large dither frequencies can be used (i.e. $\Omega \geq \Omega^*$ for a sufficiently large $\Omega^*$), whereas in a conventional model-based [89] only sufficiently small dither frequencies (i.e. $\Omega \leq \omega^*$ for a sufficiently small $\omega^*$) is allowed. While the tuning strategy in [89] results in slow convergence, the fast model-based technique may violate the actuator bandwidth in some applications. Hence, a further investigation into the sufficient conditions for the model-based approach to cover any arbitrary range of dither frequencies (i.e. for any $\Omega \in J \subset \mathbb{R}_{>0}$) would provide a significant practical advantage.

9.3.3 Further Extension of Fast Model-based ES to Multi-input Single-output plants

The analysis presented in Chapter 3 only considers the Single-Input Single-Output (SISO) plants. The tuning guidelines require that the dither frequency $\Omega$ to be tuned proportionally to $a^{-2m}$, and the optimizer adaptation gain $k$ to be tuned proportionally to $a^{2m}$, for some $m$ greater or equal to the number of unknown parameter. Therefore for sufficiently small $a$, estimating more parameter may be challenging as it demands a higher dither frequency (that can exceed the bandwidth of the actuators) and slow optimization (that contradicts the goal of fast ES). This is to be expected as more parameters are estimated.
than merely a gradient in a SISO black-box ES. However, the dimension of the estimation in a black-box ES increases with the number of inputs [82], whereas it is only a function the unknown parameters in a model-based ES. Therefore, a thorough analysis of the fast black-box ES and fast model-based ES in a Multi-Input Single-Output (MISO) ES scenario with more inputs than the unknown parameters, may reveal the merits of using the model-based approach better.

9.3.4 Using In-cylinder Pressure Transducer for Feedback

As discussed in Chapter 7, the main output feedback was the brake torque measured by a torque sensor installed on the engine–dynamometer common shaft. However, this sensor is not available in production vehicles. Therefore, it is useful to consider other alternatives of measuring the torque. In that regard, using in-cylinder pressure transducers has been considered as a promising alternative in automotive research community [34,51,86]. In-cylinder pressure measurement can be used to estimate the Indicated Mean Effect Pressure (IMEP), which directly corresponds the produced torque inside the cylinder. Therefore, the use of in-cylinder pressure transducer would provide a more realistic implementation of the proposed schemes in this thesis.

9.3.5 Considering Emission Constraints in Set-point Calibration

Emission reduction plays a major role in the adoption of alternative fuels. Therefore, considering the effect of the fuel composition on the produced emissions would be of great interest. This must be studied on the basis of different after treatment technologies such as Three Way Catalytic (TWC) converters and Selective Catalytic Reduction (SCR).

Moreover, the experimental investigations presented in Chapter 7 and 8 only consider the fuel efficiency, which can be regarded an unconstrained optimization. However, once the emission constraints are imposed, it transforms the problem to a constrained ES with some “output” constraints, where the output includes the produced emissions.
Appendix A

A.1 Proof of Theorem 3.1 - Stability

The error system is presented in equation (3.18a)-(3.18e). The following coordinate transformation is introduced for the parameter estimator error system (3.18d),

\[ \tilde{y}^2_{\pi} = \tilde{y}_0 - y^2_{\pi}, \quad \tilde{\theta}^2 = \tilde{\theta} - \theta^2. \] (A.1)

where \([y^2_{\pi}; \theta^2]\) is the solution to

\[ \begin{bmatrix} \dot{y}^2_{\pi} \\ \dot{\theta}^2 \end{bmatrix} = -\Omega A \begin{bmatrix} y^2_{\pi} \\ \theta^2 \end{bmatrix} + e(\bar{u}, a, \Omega t). \] (A.2)

Using the boundedness argument it can be shown \([y^2_{\pi}; \theta^2]\) is \(O(a\Omega^{-1})\). In the new coordinate the optimizer and the parameter estimator error equation can be written as

\[ \begin{bmatrix} \dot{y}^2_{\pi} \\ \dot{\theta}^2 \end{bmatrix} = -\Omega A \begin{bmatrix} y^2_{\pi} \\ \theta^2 \end{bmatrix} - \frac{\partial}{\partial \bar{u}} \begin{bmatrix} y^2_{\pi} \\ \theta^2 \end{bmatrix} \dot{\bar{u}} + \dot{g}_\theta, \] (A.3a)

\[ \dot{\bar{u}} = -k\Omega F(\bar{u}, \theta) + g_u + p(a, \bar{u}, \Omega t), \] (A.3b)

where

\[ g_u = k\Omega \left( F(\bar{u}, \theta) - F(\bar{u}, \theta + \bar{\theta}^2) \right), \]

\[ p(a, \bar{u}, \Omega t) = k\Omega \left( F(\bar{u}, \theta + \bar{\theta}^2) - F(\bar{u}, \theta + \bar{\theta}^2 + \theta^2) \right) \]

and \(g_\theta\) is as defined in (3.19e). In the new coordinate the non-vanishing periodic perturbation \(p(\bar{u}, a, \Omega t)\) is present in the optimizer error equation. As a result the error system
will not converge to the origin. Nevertheless, it can be shown that the trajectory of the error system will ultimately converge to a ball around the origin. Lyapunov analysis is employed to show boundedness of the solution. To that end, \( p(\bar{u}, a, \Omega t) \) is initially ignored, and asymptotic stability of the unperturbed system to the origin is investigated. Then it will be shown that the solution remains close to the origin in the presence of \( p(\bar{u}, a, \Omega t) \) for sufficiently small \( a \), and stability of this solution is preserved.

### A.1.1 Unperturbed-interconnected System

It is easy to show that the origin is a solution for the error system (3.18a)–(3.18c), (A.3a)–(A.3b) without the periodic perturbation \( p(\bar{u}, a, \Omega t) \). Moreover, without the interconnection terms, \( g_F, g_G, g_z, g_\theta \), and \( g_{ur} \), stability of the isolated subsystems can be determined by establishing stability conditions for each of them individually. This can be achieved by proposing Lyapunov functions that show asymptotic stability for each subsystem. Afterwards, a weighted sum of those Lyapunov functions will be used as a Lyapunov candidate for the interconnected system.

Considering the isolated systems, it is clear that:

- The isolated subsystem for \( \bar{x}_F, \bar{x}_G \) are Globally Exponentially Stable (GES) under Assumption 3.1 and the condition on the filters as defined in Section 3.1.2. The following Lyapunov functions can be utilized to demonstrate stability of the associated subsystems:

\[
V_F = \sqrt{\bar{x}_F^T P_F \bar{x}_F}, \quad V_G = \sqrt{\bar{x}_G^T P_G \bar{x}_G},
\]

where

\[
P_F A_F + A_F^T P_F = -I, \quad P_G A_G + A_G^T P_G = -I.
\]

- The isolated subsystems for \( \bar{z}_{G,i} \)s are GES under the condition on the filters as defined in Section 3.1.2. To capture the effect of these filters on the rest of the subsystems, a hypothetical state, \( \bar{z}_G \), is defined as

\[
\bar{z}_G = [\bar{z}_{G,1}; \bar{z}_{G,2}; \ldots; \bar{z}_{G,n_\theta}]^\theta.
\]
Therefore, \( \tilde{y}_p^T \theta \) in (3.19e) can be replaced by \( C_G \tilde{z}_G \). The state equation for \( \tilde{z}_G \) is obtained as,

\[
\dot{\tilde{z}}_G = \Omega A_G \tilde{z}_G - \frac{\partial z^*_G}{\partial \tilde{u}} \dot{\tilde{u}}, \tag{A.5}
\]

where

\[
z^*_G = \sum_{n \in \mathbb{Z} - \{0\}} (in - A_G)^{-1} B_G \hat{q}_n(\tilde{u}, a)^T e^{in\Omega \theta}. \tag{A.6}
\]

It directly follows that \( \tilde{z}_G \) is GES and the following Lyapunov functions can be used to demonstrate its stability

\[
V_Z = \sqrt{\tilde{z}_G^T P_G \tilde{z}_G},
\]

where \( P_G A_G + A_G^T P_G = -I \).

- The isolated subsystem for \( \tilde{u} \) is asymptotically stable for all initial conditions, \( \tilde{u}(t_0) \in B^1(v) \) under Assumption 3.4. Hence, the Lyapunov function \( V_u(\tilde{u}) \) can be used to establish stability of \( \tilde{u} \).

- The isolated system for \( [\tilde{y}_0^{2\pi}; \tilde{\theta}^{2\pi}] \) is globally asymptotically stable if Assumption 3.6 holds. Let \( \hat{\Theta}^{2\pi} \) represent \( [\tilde{y}_0^{2\pi}; \tilde{\theta}^{2\pi}] \). In Appendix C it is shown that there exists a Lyapunov function \( V_\Theta(t, \Theta) \) such that

\[
\begin{align*}
c_1 \| \hat{\Theta}^{2\pi} \|^2 & \leq V_\Theta(t, \hat{\Theta}^{2\pi}) \leq c_2 \| \hat{\Theta}^{2\pi} \|^2, \tag{A.7a} \\
\frac{\partial V_\Theta}{\partial t} + \frac{\partial V_\Theta}{\partial \hat{\Theta}^{2\pi}} \dot{\hat{\Theta}}^{2\pi} & \leq -c_3 a^{2n} \Omega \| \hat{\Theta}^{2\pi} \|^2, \tag{A.7b} \\
\| \frac{\partial V_\Theta}{\partial \hat{\Theta}^{2\pi}} \| & \leq c_4 \| \hat{\Theta}^{2\pi} \|, \tag{A.7c}
\end{align*}
\]

where \( c_1, c_2, c_3 \) and \( c_4 \) are \( O(1) \) positive constants.

The rate of change of the Lyapunov functions \( V_F, V_G, V_Z, V_\Theta, \) and \( V_u \) in the presence of the interconnection terms can now be studied. This problem can be approached by doing conservative analysis in which interconnection effects are investigated using their upper bounds.

**Lemma A.1.** For any given sufficiently small \( \sigma \in \mathbb{R}_{>0} \) there exists \( K_\sigma \in \mathbb{R}_{>0} \) such that (3.4b)
can be substituted with

$$-\alpha_3(|\tilde{u}|) \leq -K_{\sigma}|\tilde{u}|, \quad \forall \sigma < |\tilde{u}| < v. \quad (A.8)$$

**Remark A.1.** Lemma A.1 defines a region characterized by \(\sigma\) in which the convergence of the nominal optimizer (\(\hat{\theta} = 0\)) is upper-bounded by an exponential decay rate. In general smaller \(\sigma\) would require smaller \(K_{\sigma}\) (for example when \(\alpha_3(|\tilde{u}|) = \tilde{u}^2\)). Note that Assumption 3.4 can be strengthened to consider exponentially stable optimizers. In that case \(\sigma\) can be replaced with zero for a fixed \(K_{\sigma}\).

It can then be shown that for a given \((\sigma, K_{\sigma})\) and for sufficiently small \(a\) and \(k'\) and sufficiently large \(\Omega'\),

\[
\frac{d}{dt} \begin{bmatrix} V_F \\ V_G \\ V_Z \\ \sqrt{V_\Theta} \\ V_u \end{bmatrix} \leq A_v(k') \begin{bmatrix} \|\tilde{F}\| \\ \Omega' a^{-2m} \|\tilde{G}\| \\ \Omega' a^{-2m} \|\tilde{Z}\| \\ \sqrt{\tilde{\Theta}}^{2\sigma} \| \\ k' \Omega' |\tilde{u}| \end{bmatrix}, \quad (A.9)
\]

where

\[
A_v(k') = \begin{bmatrix} -c_{FF} & 0 & 0 & k'L_\theta c_{F\Theta} & c_{Fu} \\ c_{GF} & -c_{GG} & 0 & k'L_\theta c_{G\Theta} & c_{Gu} \\ 0 & 0 & -c_{zz} & k'L_\theta c_{z\Theta} & c_{zu} \\ c_{\Theta F} & c_{\Theta G} & c_{\Theta z} & -c_{\Theta \Theta} + k'L_\theta c_{\Theta \Theta} & c_{\Theta u} \\ 0 & 0 & 0 & k'L_\theta c_{u\Theta} & -K_{\sigma} \end{bmatrix},
\]

and \(c_{FF}, c_{F\Theta}, \ldots, c_{u\Theta}\) are all positive constants that are independent of \(a, k', \) and \(\Omega'\). Note that the inequality (A.9) is valid while \(\sigma < |\tilde{u}| < v\). Now let the Lyapunov function for the interconnected system be

\[
V = [d_F \ d_G \ d_Z \ d_\theta \ 1].[V_F \ V_G \ V_Z \ \sqrt{V_\Theta} \ V_u]^T, \quad (A.10)
\]

where \((d_F, d_G, d_Z, d_\theta) \in \mathbb{R}_{\geq 0}^4\) are yet to be determined. From (A.9), it is possible to ensure that \(dV/dt\) is negative definite by picking \(d_\theta/d_Z, d_Z/K_{\sigma}, k'L_\theta, k'L_\theta/d_\theta, d_\theta/d_G, d_G/d_F, \)
and \( d_F/K_\sigma \) to be sufficiently small. Therefore, \( V \) will decrease in time towards zero, guaranteeing convergence of \((\tilde{x}_F, \tilde{x}_G, \tilde{z}_G, \tilde{\Theta}^{2\pi}, \tilde{u})\) to the origin as long as the system states remain within the region for which (A.9) is valid. To make sure that is the case, firstly note that the weights \((d_F, d_G, d_Z, d_\theta)\) can be made arbitrary small (although this will mean \( k'L_\theta \) also has to be made small) to make \( V \to V_u(\tilde{u}) \). Secondly, from the definition of \( V \) at time \( t = 0 \) the following holds,

\[
V(0) = d_FV_F(\tilde{x}_F(0)) + d_GV_G(\tilde{x}_G(0)) + d_ZV_Z(\tilde{z}_G(0)) + d_\theta V_\theta(\tilde{\Theta}^{2\pi}(0)) + V_u(\tilde{u}(0)).
\]

Since at \( t = 0 \), \(|\tilde{u}| < r_u < v\) and \(|\tilde{\theta}^{2\pi}| < r_\theta < w\), for any given initial condition the following should hold:

\[
\begin{align*}
V(0) &\leq V_{\text{max}}(0) < \alpha_1(v), \quad \text{(A.11)} \\
V(0) &\leq V_{\text{max}}(0) < d_\theta \sqrt{c_1} w, \quad \text{(A.12)}
\end{align*}
\]

where \( V_{\text{max}}(0) \) is

\[
V_{\text{max}}(0) = d_F\sqrt{\lambda_F}\|\tilde{x}_F(0)\| + d_G\sqrt{\lambda_G}\|\tilde{x}_G(0)\| \\
+ d_Z\sqrt{\lambda_G}\|\tilde{z}_G(0)\| + d_\theta \sqrt{c_2}\|\tilde{\Theta}^{2\pi}(0)\| + \alpha_2(|\tilde{u}(0)|)
\]

where \( \lambda_F \) and \( \lambda_G \) are the largest eigenvalues of \( P_F \) and \( P_G \). Consider (A.11) and (A.12)

- for inequality (A.11) to hold, for any initial condition the weights \((d_F, d_G, d_Z, d_\theta)\) have to be made sufficiently small and \(|\tilde{u}(0)| < \alpha_2^{-1} \circ \alpha_1(v)\). If \(|\tilde{u}(0)|\) and \(\alpha_2^{-1} \circ \alpha_1(v)\) are very close then the weights (as well as \( k'L_\theta \) and \( a \)) have to be assigned very small. This ensures that even if \(|\tilde{u}|\) is to increase initially, it will not leave \( B^1(v) \);

- since \((d_F, d_G, d_Z, d_\theta)\) have to be made arbitrary small to ensure inequality (A.11), for inequality (A.12) to hold, \( w \) must be sufficiently large. This ensures that \( \tilde{\theta}^{2\pi} \) (and \( \tilde{\phi} \)) will not leave the bound characterized by \( w \). Subsequently, according to Assumption 3.5, \( L_\theta \) can be determined for that value of \( w \). Thus, \( k' \) can always be picked sufficiently small to retain the properties required for \( k'L_\theta \).

Thus, without the perturbation the trajectory converges to the origin of the error system until ultimately \(|\tilde{u}| \leq \sigma\).
A.1.2 Perturbation Effect

When the perturbation was ignored, it was established that by picking $(a, k', \Omega') \in (0, a^*) \times (0, k^*) \times (0, \Omega^*)$ for some $(a^*, k^*, \Omega^*) \in \mathbb{R}^3_{>0}$, the error system trajectory asymptotically converges to a close neighborhood of the origin. Next, it is shown that, in the presence of the non-vanishing perturbation, convergence to a small neighborhood of the origin is still achieved. To that end, observe that the rate of change of $V$ along the trajectories of the error system (3.18a)-(3.18e) satisfies

$$
\dot{V}(t, z) \leq -c_F \|\tilde{x}_F\| - c_G \Omega' a^{-2m} \|\tilde{x}_G\| - c_z \Omega' a^{-2m} \|\tilde{z}_G\| - c_\theta \Omega' \|\tilde{\Theta}^{2\pi}\|
$$

$$
- c_u k' \Omega' |\tilde{u}| + c_p L_\theta k' \Omega' \|\tilde{\Theta}^{2\pi}\|,
$$

(A.13)

where

$$
c_F = -c_{FF} d_F + c_{GF} d_G + c_{\Theta F} d_\theta, $$

$$
c_G = -c_{GG} d_G + c_{\Theta G} d_\theta, $$

$$
c_z = -c_{zz} d_Z + c_{\Theta z} d_\theta, $$

$$
c_\theta = -c_{\Theta \theta} - k' c_{\Theta \theta} d_\theta + k' c_{FG} d_F + k' c_{\Theta G} d_G + k' c_{z\theta} d_Z + k' c_{u\theta}, $$

$$
c_u = -K_\sigma + c_{Fu} d_F + c_{Gu} d_G + c_{zu} d_Z + c_{\Theta u} d_\theta, $$

$$
\Theta^{2\pi} = [y_0^{2\pi}; \theta^{2\pi}], $$

$$
c_p = [d_F d_G d_Z d_\theta 1] \left[ \frac{\partial x_F^*}{\partial u} | A_G^{-1} B_G C_F \frac{\partial x_F^*}{\partial u} + \frac{\partial x_G^*}{\partial u} | \frac{\partial z_G^*}{\partial u} | C_F \frac{\partial x_F^*}{\partial u} + | F(0) \frac{\partial \tilde{\Theta}^T}{\partial u} | \right]^T, $$

and $z$ represents $[\tilde{x}_F, \tilde{x}_G, \tilde{z}_G, \tilde{\Theta}^{2\pi}, \tilde{u}]$. To arrive at equation (A.13), the inequality $|p(\bar{u}, a, \Omega t)| < L_\theta \|\tilde{\Theta}^{2\pi}\|$ derived from Assumption 3.5 is used. The right-hand side of (A.13) is not always negative definite because, as trajectories approach the origin, the perturbation effect can make $\dot{V}$ positive. Using the ultimate boundedness notion [60], it is necessary to find a region $\Lambda = \{z|\epsilon \leq V(z)\}$ for some $\epsilon > 0$ in which

$$
\dot{V}(t, z) \leq -W(z), \quad \forall z \in \Lambda, \forall t \geq t_0,
$$

(A.14)

for some continuous positive definite function $W(z)$. In this region, the above inequality is satisfied and therefore the trajectories behave as if the origin is asymptotically stable.
According to (A.13) the region $\Lambda$ can be specified by examining each of the following inequalities which satisfy (A.14) separately:

\begin{align}
- c_F \|\dot{x}_F\| + c_p L_\theta k' \Omega' \|\Theta^{2\pi}\| < 0 \Rightarrow \|\dot{x}_F\| > \frac{c_p L_\theta k' \Omega'}{c_F} \|\Theta^{2\pi}\|, \tag{A.15a} \\
- c_G \Omega' a^{-2m} \|\dot{x}_G\| + c_p L_\theta k' \Omega' \|\Theta^{2\pi}\| < 0 \Rightarrow \|\dot{x}_G\| > \frac{c_p L_\theta k'}{c_G a^{-2m}} \|\Theta^{2\pi}\|, \tag{A.15b} \\
- c_z \Omega' a^{-2m} \|\dot{z}_G\| + c_p L_\theta k' \Omega' \|\Theta^{2\pi}\| < 0 \Rightarrow \|\dot{z}_G\| > \frac{c_p L_\theta k'}{c_z a^{-2m}} \|\Theta^{2\pi}\|, \tag{A.15c} \\
- c_\theta \Omega' \|\tilde{\Theta}^{2\pi}\| + c_p L_\theta k' \Omega' \|\Theta^{2\pi}\| < 0 \Rightarrow \|\tilde{\Theta}^{2\pi}\| > \frac{c_p L_\theta k'}{c_\theta} \|\Theta^{2\pi}\|, \tag{A.15d} \\
- c_u k' \Omega' |\tilde{u}| + c_p L_\theta k' \Omega' \|\Theta^{2\pi}\| < 0 \Rightarrow |\tilde{u}| > \frac{c_p L_\theta}{c_u} \|\Theta^{2\pi}\|. \tag{A.15e}
\end{align}

The union of the region specified in (A.15a-A.15e) can be used as the approximation of the region in which $\dot{V} < 0$. Note that $\|\Theta^{2\pi}\|$ is $O(\alpha \Omega^{-1})$. Thus inequality (A.15e) can hold for sufficiently small $\alpha$ or large $\Omega'$.

Now consider the task of finding a value of $\epsilon$ consistent with the definition of $\Lambda$. To find this $\epsilon$, first it is useful to note that

\begin{equation}
V_{\min} \leq V \leq V_{\max}, \tag{A.16}
\end{equation}

where

\begin{align*}
V_{\min} &= |d_F' d_G d_Z d_\theta 1| \times [\sqrt{\lambda_F} \|\dot{x}_F\| \sqrt{\lambda_G} \|\dot{x}_G\| \sqrt{\lambda_G} \|\dot{z}_G\| \sqrt{c_1} \|\tilde{\Theta}^{2\pi}\| \alpha_1(|\tilde{u}|)]^T, \\
V_{\max} &= |d_F' d_G d_Z d_\theta 1| \times [\sqrt{\lambda_F} \|\dot{x}_F\| \sqrt{\lambda_G} \|\dot{x}_G\| \sqrt{\lambda_G} \|\dot{z}_G\| \sqrt{c_2} \|\tilde{\Theta}^{2\pi}\| \alpha_2(|\tilde{u}|)]^T,
\end{align*}

where $\lambda_F$ and $\lambda_G$ are the smallest eigenvalues of $P_F$ and $P_G$ respectively. By using (A.15a)-(A.15e) and the RHS of inequality (A.16), it can be concluded that

\begin{equation}
\epsilon = \left( \frac{d_F \Omega' \sqrt{\lambda_F}}{c_F} + \frac{d_G \sqrt{\lambda_G}}{c_G a^{-2m}} + \frac{d_Z \sqrt{\lambda_G}}{c_z a^{-2m}} + \frac{d_\theta \sqrt{c_2}}{c_\theta} \right) \times c_p L_\theta k' \|\Theta^{2\pi}\| + \alpha_2(\sigma). \tag{A.17}
\end{equation}

Since $\|\Theta^{2\pi}\|$ is $O(\alpha \Omega^{-1})$, it follows that $\epsilon$ is $O(ak') + \alpha_2(\sigma)$ (notice the presence of $\Omega'$ in the first term in the parenthesis). That implies that by decreasing $\alpha$ and $k'$ convergence to a smaller Lyapunov surface is guaranteed.

To find the ultimate bound on the error terms, the left inequality in (A.16) can be used
to show that

$$[d_F \ d_G \ d_Z \ d_\theta \ 1] \times [\sqrt{\lambda_f}||\tilde{x}_F|| \ \sqrt{\lambda_g}||\tilde{x}_G|| \ \sqrt{\lambda_g}||\tilde{z}_G|| \ \sqrt{\epsilon_1}||\tilde{\Theta}^{2\pi}|| \ \alpha_1(||\tilde{u}||)]^T \leq \epsilon.$$ 

Since $\epsilon$ is $O(ak') + \alpha_2(\sigma)$, so is the LHS. Therefore it follows that

$$\lim_{t \to \infty} \||\tilde{x}_F(t); \tilde{x}_G(t); \tilde{z}_G(t); \tilde{y}^{2\pi}_F(t); \tilde{y}^{2\pi}_G(t); \alpha_1(||\tilde{u}(t)||)||_{\infty} = O(ak') + \alpha_2(\sigma).$$

It can be concluded that for sufficiently small $a$, $k'$, and $\sigma$ (although that requires $k^*$ to be selected accordingly), convergence of the error system states to any desired vicinity of the origin is guaranteed. Additionally, according to Remark A.1 for an exponentially stable optimizer, it is possible to replace $\sigma$ with zero. In that case, the ultimate bound is an $O(ak')$ quantity.

### A.2 Proof of Theorem 3.1 - Convergence Rate

It is a simple matter to observe that $\dot{V}$ in the region defined by the union of (A.15a- A.15d) satisfies the following inequality

$$\dot{V}(t, z) \leq -c_u k' \Omega'||\tilde{u}| + c_p L_\theta k' \Omega'||\tilde{\Theta}^{2\pi}||$$

(A.18)

For any $\delta_u \in \mathbb{R}_{>0}$, let $\epsilon_u$ such that

$$\alpha_1(\delta_u) > \alpha_2(\epsilon_u) > \alpha_1(\epsilon_u) > \alpha_2(\sigma)$$

(A.19)

From equation (A.17), it is always guaranteed that $\epsilon_u > \alpha_1^{-1} \circ \alpha_2(\sigma)$ by picking $\sigma$ to be sufficiently small. In order to make sure that once $|\tilde{u}| = \epsilon_u |\tilde{u}|$ will not increase to $\delta_u$ later on, one has to pick $d_F, d_G, d_Z, d_\theta$ as well as $k'$ sufficiently small such that $V \approx V_u$. This requires $k'$ to be tuned in $\mathcal{O}(k^{**})$ for some $k^{**} \leq k^*$. If $\epsilon_u$ is close to $\delta_u$, $k^{**}$ is expected to be less than $k^*$. Equation (A.18) can be rewritten as

$$\dot{V}(t, z) \leq -c_u k' \Omega'(|\tilde{u}| - \sigma) - c_u k' \Omega'(\sigma - \frac{c_p L_\theta}{c_u}||\tilde{\Theta}^{2\pi}||).$$

(A.20a)
A.3 Lyapunov Function for Isolated Estimator

While inequality (A.15e) holds and $|\tilde{u}| > \epsilon_u$

$$\dot{V}(t, z) \leq -c_u k' \Omega'(\epsilon_u - \sigma)$$  \hspace{1cm} (A.20b)

Now, assume that $|\tilde{u}| = \epsilon_u$ at time $t = T$. By integrating from 0 to $T$ in equation (A.18) it can be shown that,

$$V(T) - V(0) \leq -c_u k' \Omega'(\epsilon_u - \sigma) T$$  \hspace{1cm} (A.21)

since $V = d_F V_F + d_G V_G + d_Z V_Z + d_\theta V_\theta + V_u$, and therefore $V(T) \leq \alpha_2(\epsilon_u) < \alpha_1(\delta_u) \leq V(0)$ for sufficiently small $(d_F, d_G, d_Z, d_\theta, k')$, therefore $V(T) - V(0)$ is negative and

$$T \leq \frac{\alpha_1(\delta_u) - \alpha_2(\epsilon_u)}{c_u k' \Omega'(\epsilon_u - \sigma)}.$$  \hspace{1cm} (A.22)

Therefore $|\tilde{u}|$ is guaranteed to decrease to $\epsilon_u$ in an $O(1/\Omega')$ time-scale, after which $\tilde{u}$ may never increase to $\delta_u$.

A.3 Lyapunov Function for Isolated Estimator

According to Theorem 3.6.1 in [49], the isolated subsystem in (3.18d) (i.e. with $g_\theta = q(\cdot) = 0$) is GES if Assumption 3.6 holds. In order to show exponential stability, the Lyapunov function $2V_e = ||[\tilde{y}_0; \tilde{\theta}]||^2$ can be used. The time derivative of $V_e$ for the isolated estimator dynamics in the $\Omega t$ time-scale is

$$\dot{V}_e = \tilde{y}_0 \dot{\tilde{y}}_0 + \tilde{\theta}^T \dot{\tilde{\theta}} = -\left( [1; \tilde{y}_p(\tilde{u}, a, \tau)]^T \begin{bmatrix} \tilde{y}_0 \\ \tilde{\theta} \end{bmatrix} \right)^2,$$  \hspace{1cm} (A.23)

Under the persistency of excitation condition considered in Assumption 3.6, it can be shown that

$$V_e(\tilde{y}_0, \tilde{\theta}) \leq \frac{V_e(\tilde{y}_0(t_0), \tilde{\theta}(t_0))}{1 - \zeta} e^{-b(t - t_0)},$$  \hspace{1cm} (A.24a)

$$||[\tilde{y}_0(t); \tilde{\theta}(t)]||^2 \leq \frac{||[\tilde{y}_0(t_0); \tilde{\theta}(t_0)]||^2}{1 - \zeta} e^{-b(t - t_0)},$$  \hspace{1cm} (A.24b)
in which
\[ \zeta = \frac{2\alpha_0 T}{2 + \beta^4 T^2} , \quad b = \frac{1}{T} \ln \frac{1}{1 - \zeta}, \] (A.25)

\[ T = \frac{2\pi}{\Omega}, \text{ and } \beta = \sup_{\tau \geq 0} \| [1; \tilde{y}_p] \|. \] For \( \bar{u} \in \mathcal{B}(v) \) and sufficiently small dither amplitude, \( a, \| \tilde{y}_p \| \) is \( O(1) \) and therefore \( \beta \) is bounded as \( a \to 0 \).

By taking the square root of (A.24b), exponential convergence of \( [\tilde{y}_0; \tilde{\theta}] \) can be established as
\[ \| [\tilde{y}_0(t); \tilde{\theta}(t)] \| \leq \rho \| [\tilde{y}_0(t_0); \tilde{\theta}(t_0)] \| e^{-\lambda(t-t_0)} \] (A.26)

where \( \rho = \frac{1}{\sqrt{1 - \zeta}} \) and \( \lambda = b/2 \). \( \lambda \) is an important parameter since it determines how fast parameters converge. For a sufficiently small dither amplitude
\[ \lambda = \frac{1}{2T} \ln \frac{1}{1 - \zeta} = \frac{1}{2T} \ln (1 + \frac{\zeta}{1 - \zeta}) = O(a^{2m}). \] (A.27)

Similar analysis can reveal that \( \rho \to 1 \) as \( a \to 0 \).

Since \( \dot{V}_e \) is negative semi-definite it cannot be used in the interconnected system analysis. Letting \( \tilde{\Theta} \) denote \( [\tilde{y}_0; \tilde{\theta}] \) and \( \varphi(\sigma; t, \tilde{\Theta}) \) denote the solution of the isolated estimator states at time \( \sigma \) with initial conditions \( (t, \tilde{\Theta}) \), the following Lyapunov function can be proposed,
\[ V_{\tilde{\Theta}}(t, \tilde{\Theta}) = \int_{t}^{t+\delta't} \varphi^T(\sigma; t, \tilde{\Theta}) \varphi(\sigma; t, \tilde{\Theta}) d\sigma, \] (A.28)

where \( \delta' \) is a positive \( O(1) \) constant. Due to the exponentially decaying bound on the trajectories established in (A.26) and based on the Converse Lyapunov theorem [60], it can be shown that such a Lyapunov function has the following properties:

\[ c_1 \| \tilde{\Theta} \|^2 \leq V_{\tilde{\Theta}}(t, \tilde{\Theta}) \leq c_2 \| \tilde{\Theta} \|^2 \] (A.29a)

\[ \frac{\partial V_{\tilde{\Theta}}}{\partial t} + \frac{\partial V_{\tilde{\Theta}}}{\partial \tilde{\Theta}} A_{\tilde{\Theta}} \tilde{\Theta} \leq -c_3 a^{2m} \| \tilde{\Theta} \|^2 \] (A.29b)

\[ \left\| \frac{\partial V_{\tilde{\Theta}}}{\partial \tilde{\Theta}} \right\| \leq c_4 \| \tilde{\Theta} \| \] (A.29c)

where \( c_1, c_2, c_3, \) and \( c_4 \) are positive \( O(1) \) quantities.
Appendix B

B.1 Averaging Analysis for Static Map Extremum Seeking

Using the general averaging result, it is shown that the error system (5.10c) can be regarded as the averaged system (5.16) plus some perturbation terms, the effect of which can be made small. To that end first a near-identity transformation is introduced,

\[ \tilde{u}(t) =: s(t) + k \Omega p(t, s(t), k), \]  

(B.1)

where,

\[ p(t, s(t), k) = \int_0^t h(\tau, s) \exp[-k(t - \tau)]d\tau, \]  

(B.2)

\[ h(t, s) = f_{av}(s, \tilde{w}) - [f_1(s, t, \Omega t) + af_2(s, t, \Omega t)]. \]

Following the arguments in Chapter 10.6 [60], it can be shown that \( p \) and \( \partial p / \partial s \) are uniformly bounded in time and, more importantly, satisfy the following inequalities for a class \( K \) function \( \alpha(\cdot) \) and for any \( r_u < v \):

\[ k\|p(t, s, k)\| \leq \alpha(k) \quad \forall (t, s) \in [0, \infty] \times B^m(r_u) \]  

(B.3a)

\[ k\|\partial p / \partial s\| \leq \alpha(k) \quad \forall (t, s) \in [0, \infty] \times B^m(r_u) \]  

(B.3b)

In order to calculate the state equation for the new variable \( s \), one can differentiate both sides of equation (B.1) which yields

\[ \frac{d\tilde{u}}{dt} = \frac{ds}{dt} + k \Omega \frac{dp}{ds} \frac{ds}{dt} + k \Omega \frac{\partial p}{\partial t} = \frac{ds}{dt} \left( I + k \Omega \frac{\partial p}{\partial s} \right) + k \Omega (-kp + h) \]  

(B.4a)
Since \( \| \partial p / \partial s \| \) is bounded therefore and \((I + k\Omega \partial p / \partial s)\) is nonsingular therefore,

\[
\left( I + k \Omega \frac{\partial p}{\partial s} \right)^{-1} = I + O(\Omega \alpha(k)). \tag{B.5}
\]

By substituting (5.12) and (B.1) in (B.4a),

\[
\frac{ds}{dt} \left( I + k \Omega \frac{\partial p}{\partial s} \right) = -k\Omega \left[ f_1(\bar{u}, t, \Omega t) + a f_2(\bar{u}, t, \Omega t) + a^2 R \right] - k\Omega \left( -kp + h \right),
\]

\[
= -k\Omega f_{av}(s, \bar{w}) - k\Omega \left[ f_1(\bar{u}, t, \Omega t) - f_1(s, t, \Omega t) \right]
\]

\[
- k \alpha \left[ f_2(\bar{u}, t, \Omega t) - f_2(s, t, \Omega t) \right]
\]

\[
- ka^2 \Omega R + k^2 \Omega p. \tag{B.6}
\]

Now, the definition of \( s \) in (B.1) and (B.3a) can be exploited to show that

\[
f_1(\bar{u}, t, \Omega t) - f_1(s, t, \Omega t) = O(\Omega \alpha(k)), \tag{B.7a}
\]

\[
f_2(\bar{u}, t, \Omega t) - f_2(s, t, \Omega t) = O(\Omega \alpha(k)). \tag{B.7b}
\]

Therefore the state equation for \( s(t) \) is given by,

\[
\frac{ds}{dt} = -k\Omega f_{av}(s, \bar{w}) + O(\Omega k^2 \alpha(k)) + O(ka^2 \Omega) + O(ka^2 \Omega). \tag{B.8}
\]

### B.2 Proof of Theorem 5.1

The error system is presented by (5.19a) and (5.19b). It can be regarded as an interconnection of two subsystems. For ease of reference it is brought here,

\[
\hat{x} = f(\bar{x} + S, u, w(t)) - S'_u [\hat{u} + \alpha \cos(\Omega t)] - S'_w \hat{w} \tag{B.9a}
\]

\[
\hat{u} = -k\Omega h(\bar{x} + S, u) \sin(\Omega t). \tag{B.9b}
\]

To investigate stability for this system, the small gain theorem [52] is used to show that the feedback loop is practically asymptotically stable if the tuning parameters are cho-
sen appropriately. To that end, first will be shown that each subsystem is Input-State practically Stable (ISpS) (see [52]), with “input gains” that can be made arbitrarily small by tuning the parameters appropriately. This is done through the Propositions B.1 and B.2. It is noted that the style of the proof is inspired by the proof in [111], although the problem, the underlying assumptions, and to some extent the techniques are different.

**Proposition B.1.** Suppose that Assumptions 5.4-5.6 hold. Then there exists \( \beta_u \in KL \) such that for any \( \delta_u, d_u, \Delta_x \) with \( d_u < \delta_u < v \) there exist \( a^* \) and \( c^*_\gamma \) such that for any \( a \in (0, a^*) \) there exist \( \gamma_1^u, \gamma_2^u \in K_\infty \) and \( k^* \in \mathbb{R}_{>0} \) such that if \( |\tilde{u}(t_0)| < \delta_u, \|\tilde{x}\|_{t_0} < \Delta_x \) and \( (k, \Omega) \in (0, k^*) \times (0, \Omega^*) \) the following holds

\[
|\tilde{u}(t)| \leq \max \left\{ \beta_u(|\tilde{u}(t_0)|, a k \Omega (t - t_0)), \gamma^u(\|\tilde{x}\|_{t_0}), d_u \right\},
\]

where \( \gamma^u(\cdot) = \gamma_2^u (\frac{1}{a} \gamma_1^u (\cdot)) \).

**Proof.** State equation (B.9b) can be expanded as,

\[
\dot{\tilde{u}} = -k \Omega h(\tilde{x} + S(\bar{u} + a \sin(\Omega t), w(t)), \bar{u} + a \sin(\Omega t)) \sin(\Omega t)
\]

\[
= -k \Omega Q(\bar{u} + a \sin(\Omega t), w(t)) \sin(\Omega t) - k \Omega \Delta_2
\]

where

\[
\Delta_2 = [h(\tilde{x} + S(\bar{u} + a \sin(\Omega t), w(t)), \bar{u} + a \sin(\Omega t))
\]

\[
- h(S(\tilde{u} + a \sin(\Omega t), w(t)), \bar{u} + a \sin(\Omega t))] \sin(\Omega t)
\]

(B.10)

Similar to the static map case in B.1, the following transformation is used,

\[
\tilde{u}(t) =: s(t) + k \Omega q(t, s(t), k).
\]

Then by following the same steps as in Appendix A the state equation for \( s(t) \) can be obtained as,

\[
\frac{ds}{dt} = - k \Omega \frac{a}{2} Q'_u(s + u^*(\bar{w}), \bar{w})
\]
\[ k\Omega^2\alpha(k)\Delta_{1,1}(s) + ka^2\Omega\Delta_{1,2}(s) + k\Omega\Delta_2(s, \tilde{x}). \]  

(B.11)

It can be shown that on any compact set \( \Delta_{1,1}(s), \Delta_{1,2}(s), \) and \( \Delta_{1,3}(s) \) are bounded uniformly in \((a, \Omega)\), and referring to equation (B.10), \( \Delta_2(s, \tilde{x}) < \gamma_1^u(||\tilde{x}||_{t_0}) \) for some \( \gamma_1^u(\cdot) \in \mathcal{K} \) uniformly in \((a, \Omega)\). Now using the Lyapunov candidate function \( V_s(s) = |s| \) it follows that,

\[
\frac{dV_s}{dt} = -k\Omega\frac{a}{2}Q_u'(s + u^*(\tilde{w}), \tilde{w})\frac{s}{|s|} \\
+ ka^2\Omega\alpha(k)\Delta_{1,2}(s)\frac{s}{|s|} \\
+ k\Omega\Delta_2(s, \tilde{x})\frac{s}{|s|} \\
\leq -ka\left(\frac{1}{2}b|s| - \frac{1}{a}\Omega\alpha(k)\Delta_{1,1} + \Omega\alpha(k)\Delta_{1,2} \\
+ a\Delta_{1,3} - \frac{1}{a}\Delta_2(s, \tilde{x})\right),
\]

(B.12)

where \( b \) is as defined in Assumption 5.4. Using (B.12), it can be shown that

\[ |s| > \max \left\{ \frac{4}{b}\left(\frac{1}{a}\Omega\alpha(k)\Delta_{1,1} + \Omega\alpha(k)\Delta_{1,2} + a\Delta_{1,3}\right), \frac{4}{ab}\gamma_1^u(||\tilde{x}||_{t_0}) \right\}, \]

implies,

\[
\frac{dV_s}{dt} \leq -k\Omega\frac{ab}{4}|s|.
\]

(B.13)

It can be concluded that for the given initial condition, there exist strictly positive \( a^*, \Omega^* \) such that for any \( a \in (0, a^*) \) there exist strictly positive \( k^* \) such that for any \((k, \Omega) \in (0, k^*) \times (0, \Omega^*) \) and \( ||\tilde{x}||_{t_0} \leq \Delta_x, y \) is ISpS uniformly in \((a, \Omega)\) (see [52]). Since \( ||\tilde{u}|| \leq ||s|| + k\Omega||q|| \), therefore from (B.3a) it follows directly that the same conclusion holds for \( \tilde{u} \).

\[ \Box \]

**Proposition B.2.** Suppose that Assumptions 5.1-5.3 hold, then there exists \( \beta_x \in \mathcal{KL} \) such that for any \( \delta_x, d_x, \Delta_u \) with \( d_x < \delta_x \) and \( \Delta_u < \nu \), there exist \( \gamma_1^x, \gamma_2^x \in \mathcal{K}_\infty, \gamma^e \in \mathcal{K} \), and strictly positive \( a^*, k^*, \Omega^* \) such that when \( ||\tilde{x}(t_0)|| < \delta_x, ||\tilde{u}||_{t_0} < \Delta_u, \) and \((a, k, \Omega) \in (0, a^*) \times (0, k^*) \times (0, \Omega^*) \)
(0, Ω∗) and for all t ≥ t₀

\[ \|\tilde{x}(t)\| \leq \max \left\{ \beta_x(\|\tilde{x}(t₀)\|, t - t₀), \gamma^x(\|\tilde{u}\| t₀), d_x + \gamma^e(ε₁) \right\}, \]

where \( \gamma^x(\cdot) = \gamma^x_2(kΩγ^x_1(\cdot)) \).

**Proof.** The following remark is instrumental in proving Proposition B.2.

**Remark B.1.** If Assumption 5.1-5.3 hold, for fixed \( u \) and \( w \) the states of the plant will asymptotically converge to \( S(u, w) \). The results in [61] shows that under Assumptions 5.1-5.3, there exists a positive definite function \( V(\tilde{x}, u, w) \) such that,

\[
\begin{align*}
\alpha_1(\|\tilde{x}\|) &\leq V(\tilde{x}, u, w) \leq \alpha_2(\|\tilde{x}\|), \\
V_x(\tilde{x}, u, w)f(\tilde{x} + s(u, w), u, w) &\leq -\alpha_3(\|\tilde{x}\|), \\
\|V_x(\tilde{x}, u, w)\| &< b_1, \\
\|V_u(\tilde{x}, u, w)\| &< b_2, \\
\|V_w(\tilde{x}, u, w)\| &< b_3,
\end{align*}
\]

for all \( \tilde{x} \in \mathbb{R}^{N_x} \), \( u \in \mathbb{R} \), and \( w \in \Gamma \). Here \( \alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \) are \( K_\infty \) functions, and \( b_1, b_2, b_3 \) are nonnegative constants.

Next, \( V(\tilde{x}, \bar{u} + a \sin(\Omega t), w) \) is used as a Lyapunov candidate function to establish Proposition B.2. By taking the derivative of \( V \) along the trajectories of (B.9a) it is obtained that,

\[
\dot{V} = V'_x \dot{x} + V'_u \dot{\bar{u}} + V'_w \dot{w} \\
= V'_x f + (V'_u - V'_x S_u) (\dot{\bar{u}} + a\Omega \cos(\Omega t)) + (V'_w - V'_x S_w) \dot{\bar{u}} \\
\leq -\alpha_3(\|\tilde{x}\|) + (V'_u - V'_x S_u) (\dot{\bar{u}} + a\Omega \cos(\Omega t)) + (V'_w - V'_x S_w) \dot{w}. \tag{B.15a}
\]

It can be shown that for any \( \delta_x > 0 \) and \( \Delta_u < \nu \), there exist strictly positive \( a^*, \Omega^* \) and \( \gamma^x_1 \in K_\infty \), such that when \( \|\tilde{x}\| < \delta_x, \|\tilde{u}\| t₀ < \Delta_u \) and \( (a, \Omega) \in (0, a^*) \times (0, \Omega^*) \), the following
holds,

$$\dot{V} \leq -\alpha_3(\|\tilde{x}\|) + k\Omega [c_1 + c_2\alpha_3(\|\tilde{x}\|)] + \gamma_1^s(\|\tilde{u}\|_{t_0}) + a\Omega c_3 + \epsilon_1 c_4$$

for some positive constant $c_1$, $c_2$, $c_3$ and $c_4$ uniformly in $a$ and $\Omega$. Therefore for

$$\|\tilde{x}\| \geq \max \left\{ \alpha_3^{-1}(8k\Omega c_1), \alpha_3^{-1}(8k\Omega \gamma_1^s(\|\tilde{u}\|_{t_0})), \alpha_3^{-1}(8a\Omega c_3), \alpha_3^{-1}(8\epsilon_1 c_4) \right\} \quad (B.16)$$

it can be shown that $\dot{V} \leq -\frac{1}{4}\alpha_3(\|\tilde{x}\|)$. Hence, the ISpS bound of Proposition B.2 can be concluded. □

With the ISpS established for both subsystems, in the next step the proof techniques in [114] is used to show practical asymptotic stability of the system (B.9a)-(B.9b). To that end, the following analysis makes use of Propositions B.1 and B.2. It is divided into two parts. The first part deals with the stability of the system and the second part establishes practical asymptotic convergence.

**Stability**

According to Proposition B.1 and B.2 the trajectories of the system satisfy the following ISpS conditions,

$$|\tilde{u}(t)| \leq \max \left\{ \beta_u(\delta_u, a\Omega(t - t_0)), \gamma_u(\|\tilde{x}\|^t_{t_0}), d_u \right\}, \quad (B.17a)$$

$$\|\tilde{x}(t)\| \leq \max \left\{ \beta_x(\delta_x, t - t_0), \gamma_x(\|\tilde{u}\|^t_{t_0}), d_x + \gamma^e(\epsilon_1) \right\} \quad (B.17b)$$

for any $t > t_0$, if $\|\tilde{x}\|^t_{t_0} < \Delta_x$ and $\|\tilde{u}\|^t_{t_0} < \Delta_u$ and $(a, k, \epsilon_1, \Omega)$ are tuned according to the guidelines of Proposition B.1 and B.2. To prove stability for the interconnected system (B.9a)-(B.9b) it needs to be shown that

$$\|\tilde{x}\|^t_{t_0} < \Delta_x,$$

$$\|\tilde{u}\|^t_{t_0} < \Delta_u.$$
\( K \mathcal{L} \) function gives,

\[
\| \tilde{u}(t) \| \leq \max \{ \beta_u(\delta_u, 0), \gamma^u(\beta_x(\delta_x, 0)), \gamma^u(\gamma^x(\| \tilde{u} \|_{t_0})), \gamma^u(\gamma^x(\| \tilde{x} \|_{t_0})), \gamma^u(d_x + \gamma^\epsilon(\epsilon_1)), d_u \}. \tag{B.18}
\]

Since (B.18) holds for all \( t > t_0 \), the LHS can be replaced by \( \| \tilde{u}(t) \|_{t_0} \). To proceed, the following small gain condition is of use,

**Lemma B.1.** For any \( a \in \mathbb{R}_{>0} \) there exist \( k^* \) and \( \Omega^* \) such that for any \( (k, \Omega) \in (0, k^*) \times (0, \Omega^*) \) there exists \( 0 < \lambda < 1 \) and \( r_2 > r_1 > 0 \) such that \( \forall r \in [r_1, r_2], \)

\[
\gamma^u \circ \gamma^x(r) < \lambda r, \quad \gamma^x \circ \gamma^u(r) < \lambda r.
\]

**Proof.** The proof follows from the definition of \( \gamma^x \) and \( \gamma^u \).

Using Lemma B.1, (B.18) can be further simplified\(^1\) into,

\[
\| \tilde{u}(t) \|_{t_0} \leq \max \{ \beta_u(\delta_u, 0), \gamma^u(\beta_x(\delta_x, 0)), \bar{d}_u \}, \tag{B.19}
\]

where \( \bar{d}_u = \max\{\gamma^u(d_x + \gamma^\epsilon(\epsilon_1)), d_u\} \). The same steps can be repeated for \( \| \tilde{x}(t) \|_{t_0} \) which results in the following inequality under the small-gain condition of Lemma B.1,

\[
\| \tilde{x}(t) \|_{t_0} \leq \max \{ \beta_x(\delta_x, 0), \gamma^x(\beta_u(\delta_u, 0)), \bar{d}_x \}, \tag{B.20}
\]

where \( \bar{d}_x = \max\{\gamma^x(d_u), d_x + \gamma^\epsilon(\epsilon_1)\} \). The right hand side of (B.19) and (B.20) express the “maximum reach” of \( |\tilde{u}(t)| \) and \( \| \tilde{x}(t) \| \) for the given initial condition characterised by \( \delta_x, \delta_u \). Denote the maximum reach for \( \| \tilde{x} \| \) and \( |\tilde{u}| \) by \( m_x \) and \( m_u \). The following inequalities must hold to meet the conditions of Proposition B.1 and B.2,

\[
m_x \leq \min\{\Delta_x, r_2\}, \tag{B.21a}
\]

\[
m_u \leq \min\{\Delta_u, r_2\}. \tag{B.21b}
\]

In other words, the trajectories of the system described by (B.9a)-(B.9b) remain bounded\(^1\) since for any strictly positive \( a, b \) and \( 0 < \lambda < 1, a \leq \max\{\lambda a, b\} \) implies \( a \leq b \).
if (B.21a) and (B.21b) hold. In fact, referring to (B.19) and (B.20), (B.21a) and (B.21b) can be satisfied for sufficiently small \( \delta_x, \delta_u, d_u, d_x \) and sufficiently large \( \Delta_x, \Delta_u, r_2 \). This means that \((a, k, \Omega)\) need to be tuned accordingly.
B.2 Proof of Theorem 5.1

Convergence

In the second part, to prove the practical asymptotic convergence, one has to show that for any strictly positive $\tilde{d}_u$ and $\tilde{d}_x$ as defined in (B.19) and (B.20), there exists a $T > 0$ such that for all $t \geq t_0 + T$,

$$|\tilde{u}(t)| \leq \tilde{d}_u,$$

$$\|\tilde{x}(t)\| \leq \tilde{d}_x.$$

First note that, a $t_1 > 0$ can be picked such that,

$$\max \{ \beta_u(m_u, t_1), \gamma^u(\beta_x(m_x, t_1)) \} \leq \tilde{d}_u, \quad (B.22)$$

$$\max \{ \beta_x(m_x, t_1), \gamma^x(\beta_u(m_u, t_1)) \} \leq \tilde{d}_x. \quad (B.23)$$

$t_1$ is well-defined according to the definition of $KL$ functions. Now, from the conclusion of Proposition B.2 and for sufficiently small $a, k, \Omega$, for all $t \geq \frac{t_1}{ak\Omega} + t_0$ the following holds\(^2\),

$$\|\tilde{x}(t)\| \leq \max \{ \beta_x(m_x, t_1), \gamma^x(\|\tilde{u}\|_{t_0}), d_x \}.$$

Also from the conclusion of Proposition B.1 and by using the above limit for the $\|\tilde{x}\|_{\frac{t_1}{ak\Omega} + t_0}$, it can be shown that for all $t \geq \frac{2t_1}{ak\Omega} + t_0$,

$$|\tilde{u}(t)| \leq \max \{ \beta_u(m_u, t_1), \gamma^u(\beta_x(m_x, t_1)), \gamma^u(\|\tilde{u}\|_{t_0}), \tilde{d}_u \},$$

which from inequality (B.22) and Lemma B.1 can be simplified as,

$$|\tilde{u}(t)| \leq \max \{ \lambda \|\tilde{u}\|_{t_0}, \tilde{d}_u \}, \quad (B.24)$$

Repeating the same steps, reveals that for all $t \geq \frac{2jt_1}{ak\Omega} + t_0$,

$$|\tilde{u}(t)| \leq \max \{ \lambda^j \|\tilde{u}\|_{t_0}, \tilde{d}_u \}. \quad (B.25)$$

\(^2\)since it holds for all $t \geq t_1 + t_0$, it also holds for $t \geq \frac{t_1}{ak\Omega} + t_0$ for sufficiently small $ak\Omega$. 

Therefore it can be concluded that for 
\[ j \geq \frac{\log(d_u/m_u)}{\log(\lambda)} \]
and for all 
\[ t \geq \frac{2j t_1}{ak\Omega} + t_0, \]
\[ |\tilde{u}(t)| \leq \tilde{d}_u. \]  
\( (B.26) \)

Note that the time to satisfy (B.26) is scaled with \( ak\Omega \). The same steps can be repeated for 
\[ \|\tilde{x}(t)\| \]
to arrive at the same result. Hence the result of Theorem 1 can be concluded.

\section*{B.3 Sketch of Proof Of Theorem 5.2}

The proof method for Theorem 5.1 follows almost similar steps as Theorem 5.2. The difference arises firstly from the conclusion of Proposition B.1. In fact when Assumption 5.7 is used instead of Assumption 5.5, \( \tilde{u}(t) \) is required to be limited in \( B^1(v) \), as defined by Assumption 5.7 at all times. The first implication of this is that for (B.13) to hold, the amplitude of \( \|\tilde{x}\|_{t_0} \) must be picked according to the choice of \( a \). Therefore, for \( a \in (0, a^*) \), \( \Delta_x \) is picked so that \( \tilde{u}(t) \) is contained in \( B^1(v) \).

Secondly, when Assumption 5.7 is considered the condition of (B.21b) can not be guaranteed for all values of \( \epsilon_1 \). The reason is that the maximum reach of \( |\tilde{u}|, m_u \) is a function of \( \epsilon_1 \), and therefore (B.21b) can be satisfied only if \( \epsilon_1 \) is sufficiently small.

The rest of the proof is the same as Appendix B.2.

\section*{B.4 Proof of Theorem 5.3}

The system described by (5.25a)-(5.25b) exhibits discontinuity in plant states in \( t_i \) at \( \{t_1^i, t_2^i, \ldots\} \). To prove Theorem 5.3, it is shown that if the time intervals between discontinuities are sufficiently long, the ES loop can be tuned properly such that the sequence \( |\tilde{u}_i(t_i^d)| \) is ultimately bounded. The analysis in this section is analogous to the analysis of impulsive systems in [42].

To study stability for (5.25a)-(5.25b), firstly note that \( w(t) \) and the activation function \( \sigma_i(\cdot) \) govern the activation and deactivation of the ES controller and hence the sequence \( \{t_1^1, t_2^1, \ldots\} \). If \( \sigma_i(\cdot) \) satisfies Assumption 5.8, a minimum activation time of \( \tau \) is guaran-
teed, i.e.

$$t_{i_{d+1}}^d - t_i^d \geq \tau, \quad \forall d = 1, 2, \ldots$$  \hspace{1cm} (B.27)

Secondly, by using the analysis in the previous section, it is established that for any initial condition \(\tilde{u}(t_i^d), \tilde{x}(t_i^d)\) and for any strictly positive \(\kappa \geq \mu + \gamma(\epsilon_1)\), there exists a \(T\) such that \(|\tilde{u}_i(t_i)| < \kappa\) for all \(t_i > t_i^d + T\). Importantly, the analysis that leads to inequality (B.26) reveals that \(T\) is inversely proportional to \(ak\Omega\).

Now suppose \(|\tilde{u}(t_i^d)| > \mu + \gamma(\epsilon_1)\). Let \(\kappa = |\tilde{u}(t_i^d)|\). If \(\tau > T\), then it follows that,

$$|\tilde{u}(t_i^{d+1})| < |\tilde{u}(t_i^d)|.$$  \hspace{1cm} (B.28)

In other words, \(|\tilde{u}(t_i)|\) will have enough time to go below \(|\tilde{u}(t_i^d)|\) and stay there. This holds for all activations.

Now referring to the stability requirements of Theorem 5.1 in Appendix B, it is established that for a given \(a\), upper limits \(k^*\) and \(\Omega^*\) are enforced to achieve semi-global practical asymptotic stability. Suppose that \((k^*, \Omega^*)\) satisfy the conditions of Theorem 5.1. It is always possible to find a \(\tau^*\) that is larger than \(T\) produced by \(ak^*\epsilon_2^*\). In addition, for any \(\tau \in (\tau^*, \infty)\) there exist \((k^{**}, \Omega^{**}) \in (0, k^*) \times (0, \Omega^*)\) such that if \((k, \Omega) \in (k^{**}, k^*) \times (\Omega^{**}, \Omega^*)\) then it follows that \(\tau > T\), and therefore inequality (B.28) holds for all \(d = 1, 2, \ldots\), until \(|\tilde{u}_i(t_i^d)| \leq \mu + \gamma(\epsilon_1)\). It can be concluded that

$$\lim_{d \to \infty} |\tilde{u}_i(t_i^d)| \leq \mu + \gamma(\epsilon_1).$$
Bibliography


Author/s: Sharafi, Jalil

Title: Fast extremum seeking for online calibration of engines with variable natural gas composition

Date: 2016

Persistent Link: http://hdl.handle.net/11343/116672

Terms and Conditions: Copyright in works deposited in Minerva Access is retained by the copyright owner. The work may not be altered without permission from the copyright owner. Readers may only download, print and save electronic copies of whole works for their own personal non-commercial use. Any use that exceeds these limits requires permission from the copyright owner. Attribution is essential when quoting or paraphrasing from these works.